(12.5) Correlation Structures of Non stationary Data

Consider a pair of non-stationary processes $\{x(t)\}$ and $\{y(t)\}$. The mean values at any fixed time t are

$$\mu_x(t) = E[x(t)] \quad \text{and} \quad \mu_y(t) = E[y(t)]$$

The correlation functions at any pair of fixed times t_1 and t_2 are <u>defined</u> by their expected values

$$\left.\begin{aligned} R_{xx}(t_1,t_2) &= E[x(t_1)\,x(t_2)] \\ R_{yy}(t_1,t_2) &= \\ R_{xy}(t_1,t_2) &= E[x(t_1)\,y(t_2)] \end{aligned}\right\}$$

for stationary data only $(t_2 - t_1) = \tau$ mattered.

It follows that

$$R_{xx}(t_1,t_2) = R_{xx}(t_2,t_1)$$

$$R_{yy}(t_1,t_2) = R_{yy}(t_2,t_1)$$

$$R_{xy}(t_1,t_2) = R_{xy}(t_2,t_1)$$

and

$$|R_{xy}(t_1,t_2)|^2 \leq R_{xx}(t_1,t_1) \cdot R_{yy}(t_2,t_2)$$

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As before we estimate these functions by using N samples of $x_i(t)$ and compute an ensemble average

$$\hat{R}_{xx}(t_1, t_2) = \frac{1}{N} \sum_{i=1}^{N} x_i(t_1) \cdot x_i(t_2)$$

or rewrite with

$$t_1 = t \quad \text{and} \quad t_2 = t - \tau \quad (\text{or } \tau = t - t_2 = t_1 - t_2)$$

$$\hat{R}_{xx}(t, t-\tau) = \frac{1}{N} \sum_{i=1}^{N} x_i(t)\, x_i(t-\tau)$$

or rewrite with

$$t_1 = t - \tau/2 \qquad t_2 = t + \tau/2$$

then

$$\tau = t_2 - t_1 \qquad t = (t_1 + t)/2$$

time difference
lag

center time between t_1 and t_2

And

$$R_{xy}(t_1, t_2) = R_{xy}(t - \tau/2, t + \tau/2)$$

$$= E\left[x(t - \tau/2) \cdot y(t + \tau/2) \right]$$

$$= \mathcal{R}_{xy}(\tau, t)$$

For a zero lag $\tau = 0$

$$R_{xx}(\tau=0, t) = E[x^2(t)] = \sigma_x^2 \quad \text{Variance}$$

$$R_{yy}(\tau=0, t)$$

$$R_{xy}(\tau=0, t) = E[x(t)\, y(t)] = \sigma_{xy} \quad \text{Co-Variance}$$

And symetries are

$$R_{xx}(\tau, t) = R_{xx}(-\tau, t)$$

$$R_{yy}(\tau, t) =$$

$$R_{xy}(\tau, t) = R_{yx}(-\tau, t)$$

If we form an average over time, we get

$$\overline{R}_{xy}(\tau) = \lim_{T \to \infty} \int_0^T R_{xy}(\tau, t)\, dt$$

which is a real-valued, even function of τ representing the usual cross-correlation of stationary random data

(12.6) Spectral Structure of Non-Stationary Data

Assume $x(t)$ and $y(t)$ real-valued data from non-stationary processes $\{x(t)\}$ and $\{y(t)\}$ with finite Fourier Transforms

$$X(f, T) = \int_0^T x(t)\, e^{-j 2\pi f t}\, dt$$

$$Y(f, T) =$$

Dropping the record-length T notation, we define spectral density functions as

$$S_{xx}(f_1, f_2) = E[X^*(f_1) \cdot X(f_2)]$$

$$S_{yy}(f_1, f_2) =$$

$$S_{xy}(f_1, f_2) = E[X(f_1) \cdot Y(f_2)]$$

These are called "double frequency (generalized) spectral density functions." Complex for both f_1 and f_2 in the range $(-\infty, +\infty)$.

As for the time-domain correlations, we have

$$|S_{xy}(f_1, f_2)|^2 \leq S_{xx}(f_1, f_2) \cdot S_{yy}(f_1, f_2)$$

and $S'_{xx} = S'^{*}_{xx}$, $S'_{yy} = S'^{*}_{yy}$ are real + even

$S'_{xy} = S'^{*}_{yx}$ is complex for all (f_1, f_2)

Now $\underline{X}^*(f_1) \cdot \underline{Y}(f_2) = \int x(t_1) e^{-j2\pi f_1 t_1} dt_1 \cdot \int y(t_2) e^{+j2\pi f_2 t_2} dt_2$

$\Downarrow$

$$S'_{xy}(f_1,f_2) = E\left[\underline{X}^* \underline{Y}^*\right] = \iint R_{xy}(t_1,t_2)\, e^{\boxed{+j2\pi(f_1 t_1 - f_2 t_2)}} dt_1\, dt_2$$

$$\neq \iint R_{xy}(t_1,t_2)\, e^{\boxed{-j2\pi(f_1 t_1 + f_2 t_2)}} dt_1\, dt_2$$

Non-stationary Cross-Spectra NOT EQUAL double Fourier Transform of non-stationary Cross-Covariance

Instead, the non-stationary Cross-Spectra S^*_{xy} is the inverse Fourier Transform of R_{xy} followed by the Fourier Transform over t_2.

Also

$$R_{xy}(t_1, t_2) = \iint S_{xy}(f_1, f_2)\, e^{-j2\pi(f_1 t_1 - f_2 t_2)} df_1\, df_2$$

This is the direct Fourier Transform of $S'_{xy}(f_1, f_2)$ over f_1 followed by the inverse Fourier Transform over f_2.

Instead of (f_1, f_2) consider (f, g) independent frequencies

$$f_1 = f - g/2 \qquad f_2 = f + g/2$$

or

$$f = (f_1 + f_2)/2 \qquad g = f_2 - f_1$$

average frequency — difference frequency

Then

$$S'_{xy}(f_1, f_2) = E\left[X^*(f - g/2) \cdot Y(f + g/2)\right] \equiv \mathcal{S}_{xy}(f, g)$$

For $g = 0$ (that is, $f_1 = f_2$)

$$\left.\begin{aligned} \mathcal{S}_{xx}(f, 0) &= E\left[|X(f)|^2\right] \\ \mathcal{S}_{yy}(f, 0) &= \\ \mathcal{S}_{xy}(f, 0) &= E\left[X^*(f) \cdot Y(f)\right] \end{aligned}\right\}$$

standard auto-spectral and cross-spectral density for stationary processes

Further

where for $g = 0 \quad (\Rightarrow f_1 = f_2)$

$$\left.\begin{aligned} S_{xx}(f,0) &= E\left[|X(f)|^2\right] \\ S_{yy}(f,0) &= \\ S_{xy}(f,0) &= E\left[X^*(f)\,Y(f)\right] \end{aligned}\right\}\ \text{standard auto- and cross-spectral density functions for stationary processes}$$

Further development shows that

$$\underbrace{\int R_{xy}(\tau,t)\, e^{-j2\pi f\tau}\, d\tau}_{\substack{\text{Fourier Transform of } R_{xy}(\tau,t) \\ \text{over } \tau \text{ holding } (f,t) \text{ fixed}}} = \underbrace{\int S_{xy}(f,g)\, e^{+j2\pi g t}\, dg}_{\substack{\text{Inverse Fourier Transform of} \\ S_{xy} \text{ over } g \text{ holding} \\ (f,t) \text{ fixed}}}$$

recall

$$t = (t_1 + t_2)/2 \qquad f = (f_1 + f_2)/2$$

$$\tau = t_2 - t_1 \qquad g = f_2 - f_1$$

Either of these operations defines a $\boxed{\text{time-frequency function}}$

$$W_{xy}(f,t) = \int R_{xy}(\tau,t)\, e^{-j2\pi(f\tau)}\, d\tau$$

Do NOT confuse ~~with~~

with $\quad S_{xy}(f,g) = \iint R_{xy}(\tau,t)\, e^{-j2\pi(f\tau + gt)}\, d\tau\, dt$

that is the double Fourier Transform of $R_{xy}(\tau,t)$

For stationarity data, we have

$$R_{xy}(t_1, t_2) = R_{xy}(t_2 - t_1) = R_{xy}(\tau)$$

and

$$R_{xy}(t, \tau) = R_{xy}(\tau)$$

Consider the non-stationary

(p.-180) $$R_{xy}(t,t) = \iint S_{xy}(f_1, f_2)\, \underbrace{e^{-j2\pi t (f_1 - f_2)}}\, df_1\, df_2$$

This now becomes

$$R_{xy}(t,t) \underset{\substack{\uparrow \\ \text{stationarity}}}{=} R_{xy}(0) = \iint S_{xy}(f_1, f_2) \cdot \overbrace{1}^{t=0}\, df_1\, df_2$$

$$= \int S_{xy}(f_1)\, df_1$$

where
$$S_{xy}(f_1) = \int S_{xy}(f_1, f_2)\, df_2$$

Hence for stationary data we have

$$S_{xy}(f_1, f_2) = S_{xy}(f_1) \cdot \delta_1(f_2 - f_1)$$

where
$$\delta_1(f) = \begin{cases} T & -1/2T \le f < 1/2T \\ 0 & \text{otherwise} \end{cases}$$

Frequency - Time Spectra

$$\mathcal{R}_{xy}(\tau, t) = E\left[x(t-\tau/2)\ \ y(t+\tau/2)\right]$$ non-stationary cross-covariance

and

$$\mathcal{S}_{xy}(f, g) = E\left[X^*(f-g/2) \cdot Y(f+g/2)\right]$$ non-stationary cross-spectra

The Fourier Transform of $\mathcal{R}_{xy}(\tau, t)$ with respect to τ holding t fixed gives

$$\mathcal{W}_{xy}(f, t) = \int \mathcal{R}_{xy}(\tau, t)\, e^{-j2\pi f\tau}\, d\tau$$

Defines the frequency-time spectral density which is also called the "instanteneous spectrum"

Note that

$$\mathcal{W}_{xy}(f, t) = \int \mathcal{S}_{xy}(f, g)\, e^{+j2\pi g t}\, dg$$

So the frequency-time spectra is also the inverse Fourier Transform of $\mathcal{S}_{xy}(f, g)$ with respect to g while holding f constant.

The reverse holds, too

$$\mathcal{R}_{xy}(\tau, t) = \int \mathcal{W}_{xy}(f, t)\, e^{j 2\pi f \tau}\, df$$

and/or

$$\mathcal{S}_{xy}(f, g) = \int \mathcal{W}_{xy}(f, t)\, e^{-j 2\pi g t}\, dt$$

$\Updownarrow$

$$\mathcal{S}_{xy}(f, g) = \iint \mathcal{R}_{xy}(\tau, t)\, e^{-j 2\pi (f\tau + gt)}\, d\tau\, dt$$

$$\mathcal{R}_{xy}(\tau, t) = \iint \mathcal{S}_{xy}(f, g)\, e^{+j 2\pi (f\tau + gt)}\, df\, dg$$

are Double Fourier Transform Pairs.

Special cases occur for non-stationary auto-correlations $\mathcal{R}_{xx}(\tau, t)$ for which

$$\mathcal{W}_{xx}(f, t) = \int R_{xx}(\tau, t)\, e^{-j 2\pi f \tau}\, d\tau = \int R_{xx}(\tau, t) \cos(2\pi f \tau)\, d\tau$$

This is a real and even function $\mathcal{W}_{xx}(f, t) = \mathcal{W}^{*}_{xx}(f, t)$

$$\mathcal{W}_{xx}(f, t) = \mathcal{W}_{xx}(-f, t)$$

And for $\tau = 0$

$$\mathcal{R}_{xx}(\tau = 0, t) = \int \mathcal{W}_{xx}(f, t)\, df = E\left[x^2(t)\right]$$

$$= \sigma_x^2(t)$$