Wavelets and Signal Processing

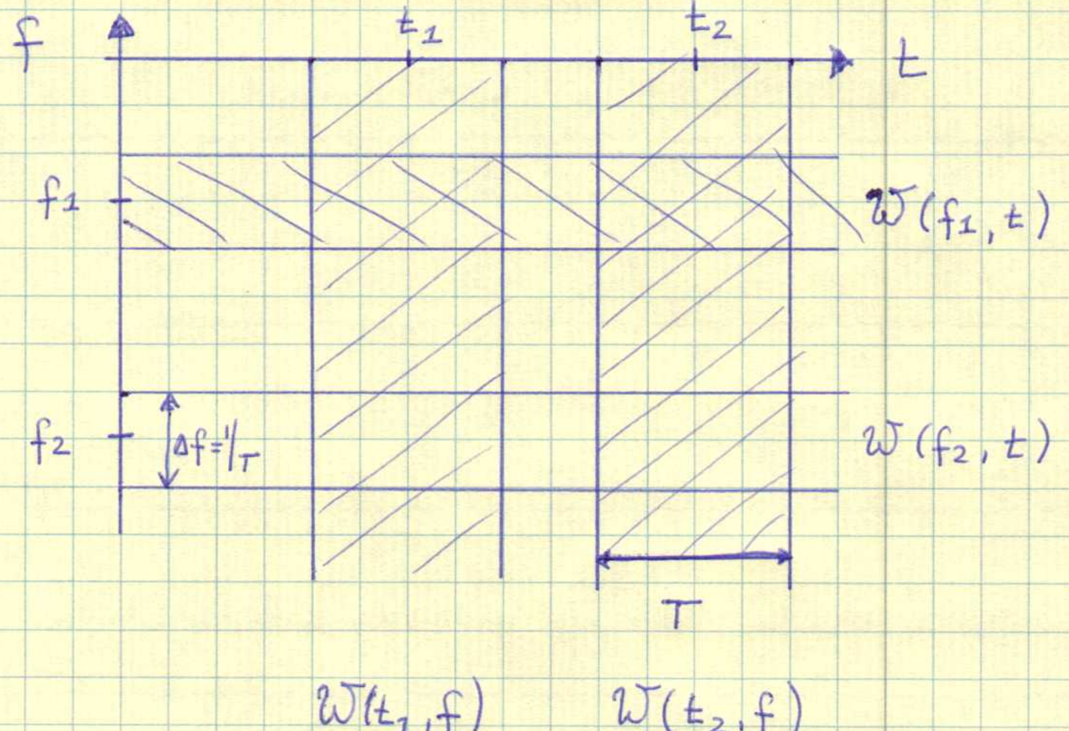
As an alternative to Short-Time Fourier Transform

or Windowed Fourier Transform Methods

$$W_{xx}(f,t) = \int R_{xx}(\tau,t)\, e^{-j 2\pi f \tau}\, d\tau$$

which is the Fourier Transform of $R_{xx}(\tau,t)$ performed on a sliding segment or window of length T. The frequencies f then are a sequence $\frac{1}{T}, \frac{2}{T}, \frac{3}{T}, \dots, 1/2\Delta t = \frac{N/2}{T}$

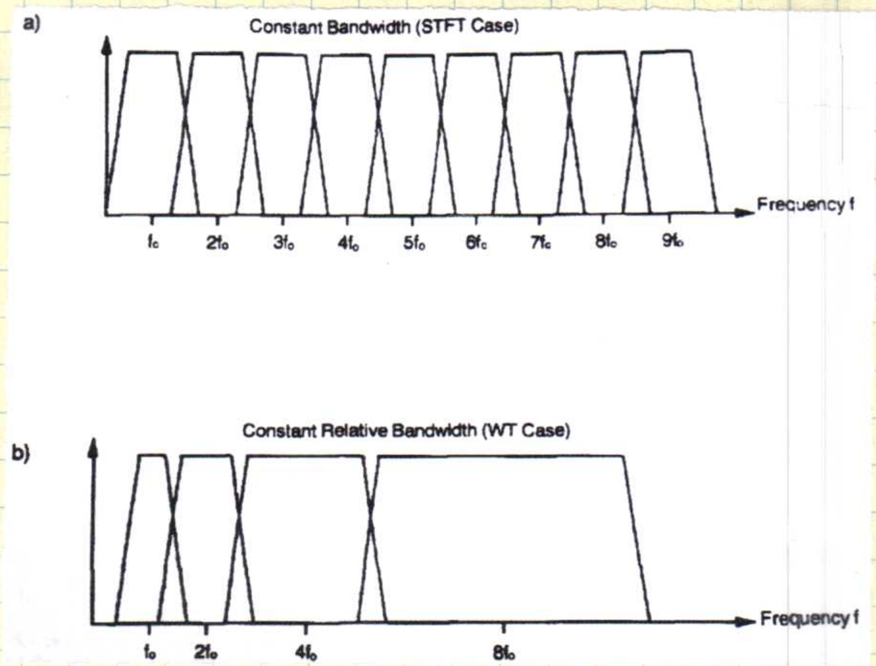
at each time t with $T = N \cdot \Delta t$



Note that both the time and frequency axes consist of discrete set of windows each of length (time-resolution) T and (frequency resolution) $1/T$ & $\Delta f \cdot \Delta T = 1$

The time-frequency localization is said to depend on the scale T that in Fourier Analyses must be chosen a priori. We always for each short window or data segment have N/2 discrete frequencies at which we evaluate our data.

The wavelet analysis removes this stringent scale-T dependence as it replaces the fixed bandwidth $\Delta f = 1/T = \text{const.}$ with a bandwidth $\Delta f = \Delta f\,(f)$ that varies with frequency:



Short Fourier Transform

$\Delta f = \text{const.}$

Wavelet Transform

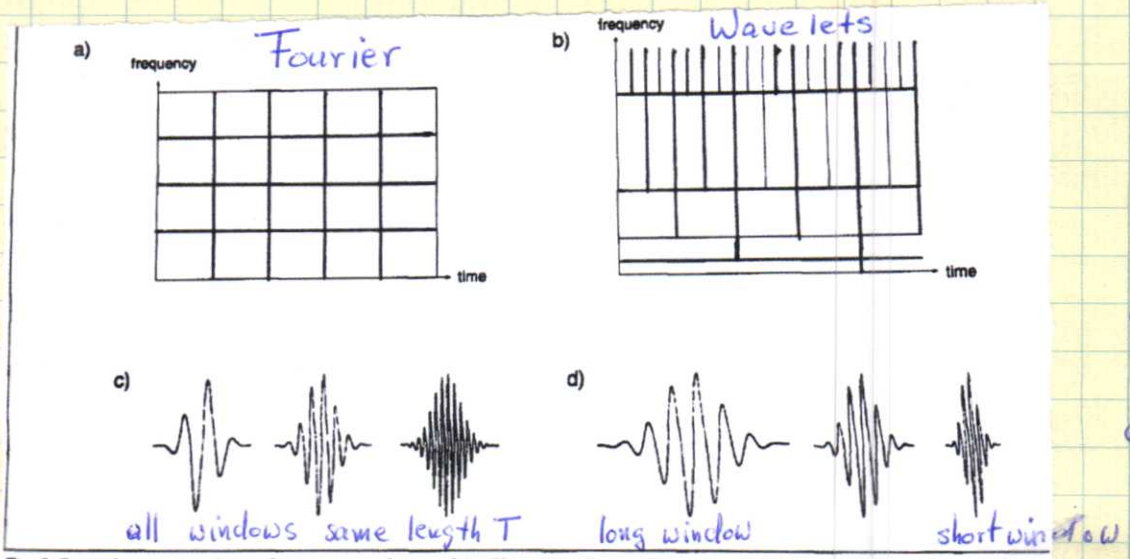
$\Delta f = \Delta f\,(f)$

narrow band — wide band, short T

long T — short T

for long periods — for short periods

So the time-frequency or time-scale domain looks like



Resolution in (t, f) plane

Base functions at

Fig. 2. Basis functions and time-frequency resolution of the Short-Time Fourier Transform (STFT) and the Wavelet Transform (WT). The tiles represent the essential concentration in the time-frequency plane of a given basis function. (a) Coverage of the time-frequency plane for the STFT, (b) for the WT. (c) Corresponding basis functions for the STFT, (d) for the WT ("wavelets").

Fourier $\Delta f = \text{const}$

Wavelet $\Delta f = \text{const.} \cdot f$ or $\dfrac{\Delta f}{f} = \text{const}$

The Continuous Wavelet Transform (CWT) conserves

$$\Delta f(f) \cdot T(f) = 1$$

but unlike the Fourier Transform both Δf and T depend on frequency. This is accomplished with a scaled (stretched or compressed) version of the same prototype wavelet $\psi(t)$, e.g.,

$$\psi_a(t) = \frac{1}{\sqrt{|a|}} \psi(t/a)$$

where a is the scale factor

The wavelet transform of data $x(t)$ is then defined as a function of time "τ" and scale "a" as

$$CWT_x(\tau, a) = \frac{1}{\sqrt{|a|}} \int x(t)\, \psi^*\left(\frac{t-\tau}{a}\right) dt$$

The wavelet analysis is self-similar at all scales (think of these as frequency)

One can also think of the basic wavelet $\psi(t)$ as a "modulated data window"

$$\psi(t) = g(t)\, e^{-j 2\pi f_0 t}$$

in the context of Fourier Analysis. Here $g(t)$ is the modulation.

Note that if a function $x(t)$ is scaled

$$x(t) \longrightarrow x(a \cdot t) \qquad a > 0$$

then it is contracted if $a > 0$

and expanded if $a < 0$

Thus the CWT can be written also as

$$CWT_x(\tau, a) = \sqrt{a} \int \frac{x(a \cdot t)}{\cancel{x(ta)}} \cdot \psi(t - \tau/a)\, dt$$

via a change of variables

Morlet Transform

An admissable wavelet must have zero mean and be localized in both time and frequency space. A plane wave modulated by a Gaussian is one such example

$$\Psi(t) = \pi^{-1/4} e^{j\omega_0 t} e^{-t^2/2}$$

where ω_0 is a non-dimensional frequency.

For a discrete set of data $x_i = x(i\Delta t)$ $i=1,2,\dots N$ the continuous Wavelet Transform becomes

$$\boxed{W_i(a) = \sum_{k=0}^{N-1} x_k \, \Psi^* \underbrace{\left(\frac{(k-i)\Delta t}{a}\right)}_{\text{argument of wavelet } \Psi^*}}$$

complex conjugate

The scale "a" of the wavelet can be varied

The $W_i(a)$ is a convolution of the data x_i with a scaled and translated version of $\Psi(t)$ for which

$$t = \frac{k-i}{a} \cdot \Delta t$$

Convolution in time is a multiplication in frequency

with discrete Fourier Transform of $x(t)$ and $\Psi(t)$

denoted as $$\hat{x}_k = \frac{1}{N} \sum_{i=1}^{N-1} x_i \, e^{-j 2\pi i k/N}$$

and $\Psi(t/a) \longrightarrow \hat{\Psi}(a \cdot \omega)$ in the frequency domain

Then the wavelet transform is the inverse Fourier Transform of the product

$$\boxed{W_i = \sum_{k=0}^{N-1} \hat{x}_k \cdot \hat{\Psi}^*(a\omega_k) \, e^{+j \omega_k \cdot i \cdot \Delta t}}$$

where the angular frequency

$$\omega_k = \begin{cases} 2\pi k / N \cdot \Delta t & k \leq N/2 \\ -2\pi k / N \cdot \Delta t & k > N/2 \end{cases}$$

$2\pi f_k = \omega_k$

$$f_k = \begin{cases} k / T & k \leq N/2 \\ -k / T & k > N/2 \end{cases}$$

So, practically speaking, the best and fastest way to get the wavelet transform is to the standard discrete Fourier Transform on data $x(t)$ and wavelet $\Psi(t/a)$ for a set of "scales" a !

Normalization of Wavelets $\longrightarrow$ Wavelet Power Spectra

Recall that the "scale a" ~~is~~ acts like a frequency in the Short Fourier Transform of time-frequency spectra.

To compare the respective wavelet transforms for a given scale "a" with another scale "2·a", say, we need to normalize to have unit ~~energy~~ variance.

$$\hat{\Psi}(a \cdot \omega_k) = \frac{2\pi a}{\Delta t} \hat{\Psi}_0 (a \cdot \omega_k)$$

For Morlet $\Psi_0(t) = \pi^{-1/4} e^{j\omega_0 t} e^{-t^2/2}$

$$\hat{\Psi}_0 (a \cdot \omega) = \pi^{-1/4} e^{-(a\omega - \omega_0)^2/2} \cdot \begin{cases} 1 & \omega > 0 \\ 0 & \omega \leq 0 \end{cases}$$

and $$\int_{-\infty}^{+\infty} \left| \hat{\Psi}_0(\omega') \right|^2 d\omega' = 1$$ normalized to have unit variance

At each scale "a" we have

$$\sum_{k=0}^{N-1} \left| \hat{\Psi}(a \cdot \omega_k) \right|^2 = N$$

193

For the convolution $W_i(a)$ on page-190 the normalization is

$$\psi\left[\frac{(k-i)}{a}\Delta t\right] = \left(\frac{\Delta t}{a}\right)^{1/2} \psi_0\left[\frac{(k-i)}{a}\Delta t\right]$$

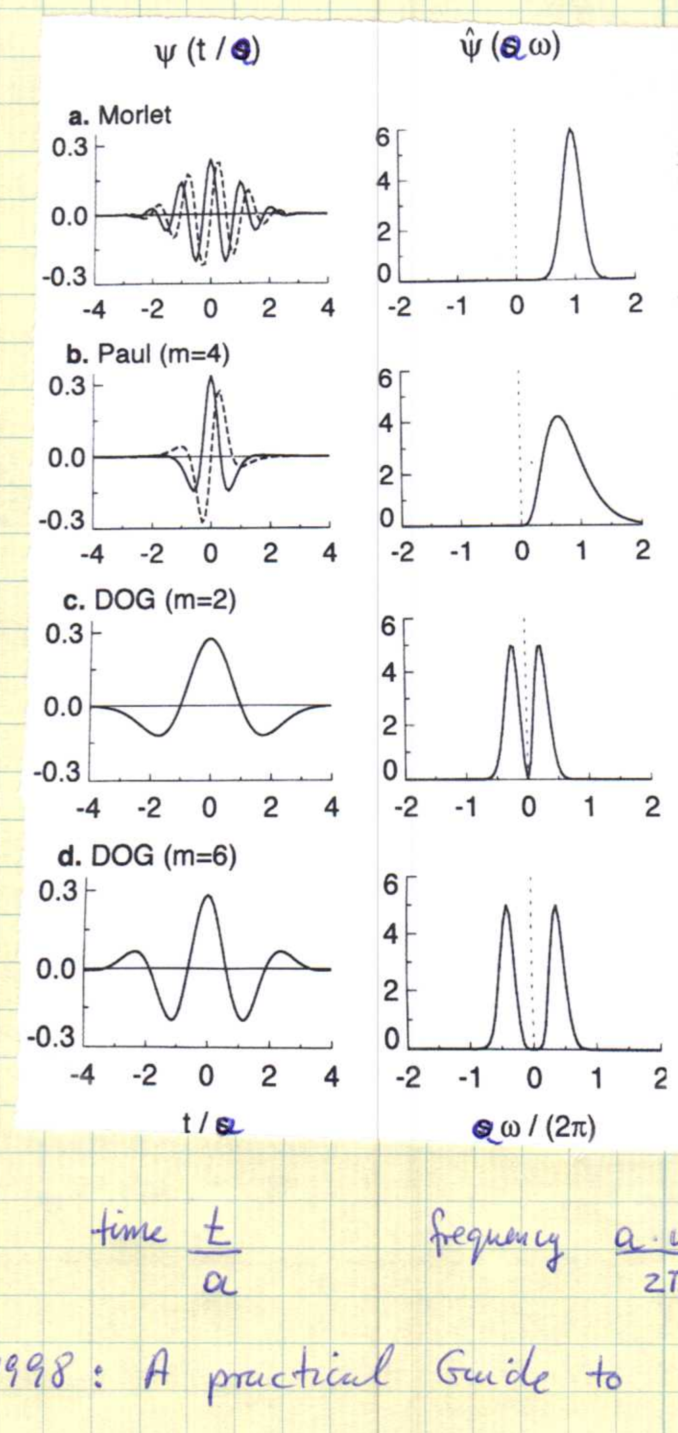
Some non-orthogonal Wavelets:

Morlet Transform (complex)

Paul Transform (complex)

Derivative of Gaussian (real)

Derivative of Gaussian (real)



time $\frac{t}{a}$ frequency $\frac{a \cdot \omega}{2\pi}$

Torrence, C. and G. P. Compo, 1998: A practical Guide to Wavelet Analysis. Bull. Americ Met. Soc., 79, 61–78.

Wavelet Power Spectrum

The wavelet transform $W_i(a) \hat{=} W(t_i, a) \hat{=} W(t, f)$ is generally complex and thus has amplitude $|W_i(a)|$ and phase $\tan^{-1}[\,\mathrm{Im}(W_i(a)) / \mathrm{Re}(W_i(a))]$.

Thus we define the power spectrum as

$$|W_i(a)|^2 = W_i(a) \cdot W_i^*(a)$$

With proper normalization the expected value

$$E\left[|W(t_i,a)|^2\right] = N \cdot E\left[|\hat{x}_\ell|^2\right] = \cancel{N}\frac{\sigma^2}{\cancel{N}}$$

white noise

where σ^2 is the variance.

Thus, for a white noise random process $\{X(t)\}$

$$E\left[|W(t_i,a)|^2\right] = \sigma^2$$ for all t_i and all a

Choice of Scales "a"

For orthogonal wavelets, one must choose a discrete set

For non-orthogonal wavelets (such as Morlet) anything goes including

$$a_j = a_0\, 2^{\ell \cdot \Delta \ell} \qquad \ell = 0, 1, 2, \dots L$$

$L = \log_2(N \cdot \Delta t / a_0)$

Choice of Scales "a"

For orthogonal wavelets, one must choose a discrete set

For non-orthogonal wavelet (such as the Morlet Transform) anything goes and includes

$$a_l = a_0 \, 2^{l \cdot \Delta l} \qquad l = 0, 1, 2, \dots L$$

where

$$L = \log_2 \left(N \cdot \Delta t / a_0 \right)$$

Choose a_0 to be close to about $(2 \cdot \Delta t)$ as in $f_{Nyquist} = 1/2\Delta t$

Wavelet Scale "a" and Fourier Frequency "f"

Note that Figure on page-193 does not always show a peak for $\hat{\Psi}(a \cdot \omega)$ at $\frac{a\,\omega}{2\pi} = a \cdot f = 1$

For the Morlet Wavelet with $\omega_0 = 6$ $\quad \frac{1}{f} = 1.03\, a$

both the scale "a" and the period "f^{-1}" are close, but this is NOT ALWAYS the case.

→ always calculate "equivalent Fourier period" for each scale "a".

How to do this?

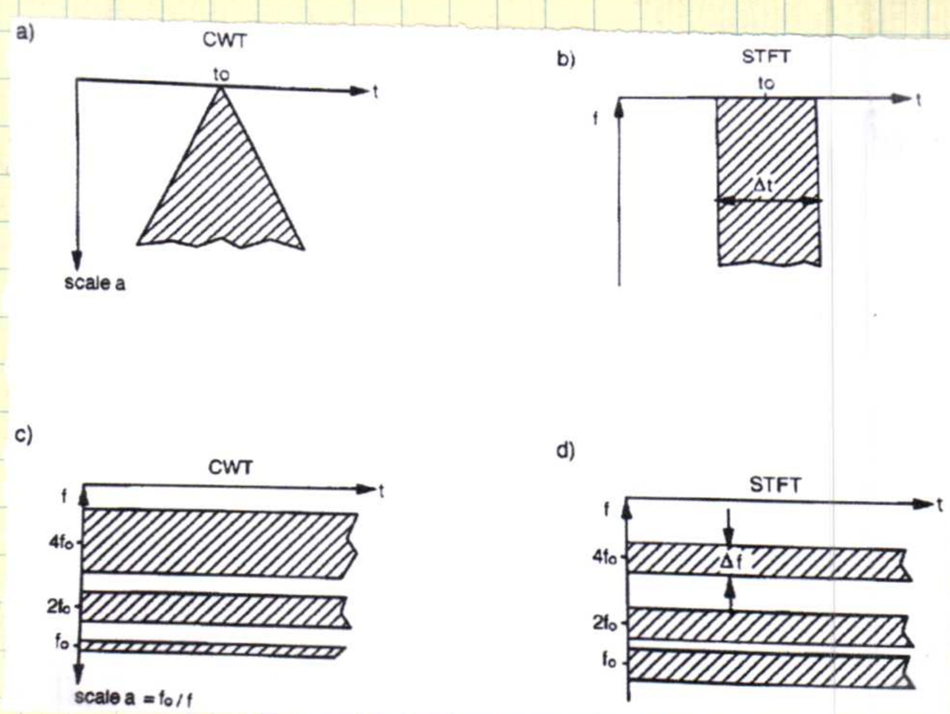
Recall

$$W_i(a) = \underbrace{\sum_{k=0}^{N-1}}_{\text{sum over all frequencies}} \overbrace{\hat{x}_k}^{\text{FT of data}} \cdot \underbrace{\hat{\psi}^*_k(a \cdot \omega_k)}_{\text{FT of wavelet}} \underbrace{e^{+j\,\omega_k\, i \cdot \Delta t}}_{\text{inverse Fourier Transform}}$$

Use a known cosine of a known frequency for $x(t)$ and $\hat{x}_k(f_k)$ and then search for the wavelet power spectrum maximum at scale "a".

→ Always convert *scale "a" to Fourier Period before plotting

Wavelet | Short Time Fourier | Regions of Influence:



Dirac δ-function

cosines at f_0, $2f_0$, $4f_0$

Rioul, O. and M. Vetterli, 1991: Wavelets and signal processing. IEEE Signal Proc. Mag., Vol. 8, p. 14-38.