1.4 Group Velocity

velocity of a wave packet
constructed by superposition

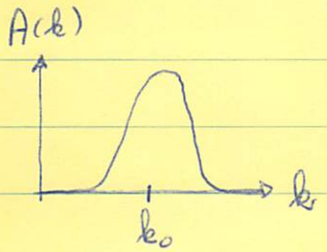
Assume we have some

$$\sigma = \sigma(k) \qquad \text{dispersion relation}$$

Superposition means

$$\phi(x,t) = \int \underbrace{A(k)}_{\text{amplitudes}} \; \underbrace{e^{i[kx - \sigma(k)t]}}_{\text{phase } \varphi = kx - \sigma t}$$

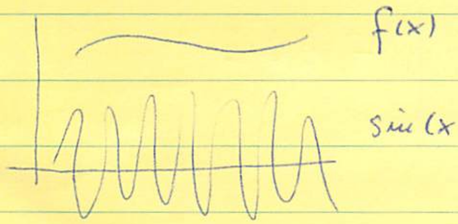
wave packet :



How does this wave packet move ?

<u>Method of Stationary Phase</u>

If you have an integral of a rapidly oscillating and a slowly varying function, then you get something close to zero:



$$\int f(x)\,\sin(x)\,dx \approx 0$$

because the "+" cancels the "−"

→ need slowly varying phase to get contributions to the integral near k_0

So we need

$$\varphi(k) = kx - \sigma(k)\cdot t$$

to become "stationary", that is, to vary very little :

$$\frac{d\varphi}{dk} = x - \left.\frac{d\sigma}{dk}\right|_{k_0} \cdot t = 0 \; ?$$

↳ $$x = \boxed{\left.\frac{d\sigma}{dk}\right|_{k_0}} \cdot t$$

group velocity $\left. c_g \right|_{k_0} = x / t$

Next we need to show that the wave pattern $\phi(x,t)$ indeed moves at this speed. For this we need to solve the integral which at $t=0$ is #3

$$\phi(x, t=0) = \int A(k)\, e^{ikx}\, dk$$

$$\phi(x,t) = \int A(k)\, e^{i(kx - \sigma t)}\, dk$$

All we know is that our wave packet peaks near k_0

↳ $$\sigma(k) = \sigma(k_0) + (k-k_0)\left.\frac{d\sigma}{dk}\right|_{k_0} + O\left((k-k_0)^2\right)$$

Taylor Expansion

So

$$\phi(x,t) = \int A(k) e^{ikx} \underbrace{e^{-i\sigma(k_0)t}} e^{-ik \frac{d\sigma}{dk}\big|_{k_0} t} \underbrace{e^{+ik_0 \frac{d\sigma}{dk}\big|_{k_0} t}} e^{\text{negligible}} \, dk$$

$$\phi(x,t) = \underbrace{e^{-it\left[\sigma(k_0) - \frac{d\sigma}{dk}\big|_{k_0} t\right]}} \cdot \underbrace{\int A(k)\, e^{ik\left[x - \frac{d\sigma}{dk}\big|_{k_0} \cdot t\right]} \, dk}$$

this is just a pure phase whose magnitude is always 1

this "looks" like $\phi(x', t=0)$ with

$$x' = x - \frac{d\sigma}{dk}\bigg|_{k_0} \cdot t$$

$$|\phi(x,t)| = 1 \cdot \phi\left(x - \frac{d\sigma}{dk}\bigg|_{k_0} \cdot t \,,\ 0\right)$$

The wave packet @ time t is the same as at time $t=0$

but move ~~to~~ from x to $\left(x - \frac{d\sigma}{dk}\Big|_{k_0} \cdot t\right)$