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Whitman
p. 369 ff.

$$\eta(x,t) = \int_{-\infty}^{+\infty} A(k)\, e^{ikx - i\sigma(k)t}\, dk$$

For wave motions consider behavior for both large $x$ and $t$

or $x/t$ fixed with $t \to \infty$ (asymptotic behavior).

Rewrite

$$\eta(x,t) = \int_{-\infty}^{+\infty} A(k)\, \underbrace{e^{-i\Theta t}}_{\text{oscillation}}\, dk$$

Phase $\Theta(k)$:

$$\Theta(k) = \sigma(k) - \frac{kx}{t}$$

Evaluate this integral with "stationary phase"

Main contributions to integral come from the neighborhood $k = k_0$

where $\left.\frac{d\Theta}{dk}\right|_{k=k_0} = 0$

$$\left.\frac{d\Theta}{dk}\right|_{k=k_0} = \left.\frac{d\sigma}{dk}\right|_{k=0} - \frac{x}{t}$$

$$\Rightarrow \left.\frac{d\sigma}{dk}\right|_{k=k_0} = \frac{x}{t}$$

(or condition to fix $k_0$)

With this additional (new, asymptotic) behavior expand ~~the~~

$A(k)$ and $\Theta(k)$ as Taylor series near $k = k_0$

$$A(k) \approx A(k_0) \quad \text{and} \quad \Theta(k) = \Theta(k_0) + \underbrace{(k-k_0)\left.\frac{d\Theta}{dk}\right|_{k=k_0}}_{=0} + (k-k_0)^2 \left.\frac{d^2\Theta}{dk^2}\right|_{k=k_0} + \dots$$

(1) $\sigma = (g k)^{1/2}$     Deep Water Wave Dispersion

(2) $\theta = k \frac{x}{t} - (g k)^{1/2}$     Phase, e.g., $e^{i(kx - \sigma t)} = e^{i\theta t}$

oscillation

(3) $\frac{d\theta}{dk} = \frac{x}{t} - \frac{1}{2}\left(\frac{g}{k}\right)^{1/2}$     $\frac{d\sigma}{dk} = \frac{1}{2}\left(\frac{g}{k}\right)^{1/2}$ group velocity

(4) $\left.\frac{d\theta}{dk}\right|_{k=k_o} = 0$     Stationary Phase ( $x/t$ equals wave group vel. )

$\Rightarrow \frac{x}{t} = \frac{1}{2}\left(\frac{g}{k_o}\right)^{1/2}$ or $\frac{x^2}{t^2} = \frac{1}{4}\frac{g}{k_o}$

$\Rightarrow k_o = \frac{g}{4}\frac{t^2}{x^2}$

$\Rightarrow \theta(k_o) = \frac{g}{4}\frac{t^2}{x^2}\cdot\frac{x}{t} - \left(g\,\frac{g}{4}\frac{t^2}{x^2}\right)^{1/2} = -\frac{1}{4}\,g t/x$

(5) $\left.\frac{d^2\theta}{dk^2}\right|_{k=k_o} = \frac{d}{dk}\left[\frac{x}{t} - \frac{1}{2}\left(\frac{g}{k}\right)^{1/2}\right] = -\frac{1}{4}\left(\frac{g}{k}\right)^{-1/2}\cdot(-)\frac{g}{k^2}$

$$= +\frac{1}{4}\, g^{1/2}\, k^{-3/2} = \frac{g^{1/2}}{4}\cdot\underbrace{\frac{8x^3}{g^{3/2}t^3}}_{k_o^{-3/2}} = \frac{2x^3}{g t^3}$$

(6) $$\eta(x>0,t) = \frac{1}{2} A(k_0)\, e^{+i\Theta(k_0)\cdot t} \cdot \left(\pi/\alpha\right)^{1/2}$$

(z=k-k0 and dz=dk)

$$\int_{-\infty}^{+\infty} e^{-\alpha z^2}\, dz = \left(\frac{\pi}{\alpha}\right)^{1/2} : \qquad \alpha = -it\, \Theta''(k_0)/2$$

$$= -it\, \frac{2x^3}{g t^3} \cdot \frac{1}{2} = -\frac{i x^3}{g t^2}$$

$$\Downarrow \quad \eta(x>0,t) = \frac{1}{2} A(k_0)\, e^{i(-)gt^2/4x} \cdot \left(\pi \cdot \frac{gt^2}{-ix^3}\right)^{1/2}$$

$$= \frac{1}{2} A(k_0)\, e^{-igt^2/4x} \cdot e^{-i\pi/4} \cdot \left(\frac{\pi g t^2}{x^3}\right)^{1/2}$$

$$\boxed{\begin{aligned} i &= e^{i\pi/2} \\ i^{-1/2} &= e^{i\pi/2\cdot(-)1/2} \\ &= e^{-i\pi/4} \end{aligned}}$$

(7) Same line of reasoning for $k<0$ gives

$$\eta(x<0,t) = \eta(x>0,t) \quad \Downarrow \quad \eta(x,t) = A(k_0)\, e^{-i(gt^2/4x + \pi/4)} \left(\frac{\pi g t^2}{x^3}\right)^{1/2}$$

(8) $\eta(x,t=0) = \delta(x) \quad \Downarrow \quad A(k_0) = 1/2\pi$

$$\Downarrow \quad \eta(x,t) = \frac{1}{2\pi}\, e^{-i(gt^2/4x + \pi/4)} \cdot \left(\frac{\pi g t^2}{x^3}\right)^{1/2}$$

In 1-D

$$\eta(x,t) = \frac{1}{2\pi}\, e^{-i\pi/4}\, e^{-i\, g t^2/4x} \cdot \frac{(4\pi)^{1/2}}{x} \left(\frac{g t^2}{4x}\right)^{1/2}$$

$$= \frac{(4\pi)^{1/2}}{2\pi}\, e^{-i\pi/4}\, \frac{e^{-iP}}{\cancel{3}} \cdot \frac{P^{1/2}}{x}$$

$$= \frac{1}{\sqrt{\pi}}\, e^{-i\pi/4}\, e^{-iP}\, \frac{P^{1/2}}{x}$$

Text

$$= \boxed{K_1 \cdot \frac{P^{1/2}}{x} \cdot e^{-iP}} \qquad P = \frac{g t^2}{4x}$$

in 2-D

$$\eta(x,y,t) = \boxed{K_2 \left[\frac{P^{1/2}}{r}\right]^2 e^{-iP}} \qquad P = \frac{g t^2}{4r} \qquad r = (x^2+y^2)^{1/2}$$

$$\frac{200,000}{320\ \text{mil}} = .625$$