$\boxed{\text{Case 2:}}$ $N \neq 0$, $f = 0$ $\qquad |c_g| = \frac{|N^2 - \cancel{f^2}|}{\sigma \kappa} \sin\varphi' \cos\varphi'$

$$\sigma = N \cos\varphi' \qquad |c_g| = \frac{|N^2| \sin\varphi' \cos\varphi'}{\kappa N \cos\varphi'}$$

$$= |N| \sin\varphi' / \kappa$$

$$\vec{c}_g \propto \underbrace{\cancel{(N^2 - \sigma^2)}}_{N^2(1-\cos^2\varphi')} \begin{pmatrix} k \\ l \\ m \end{pmatrix} - N^2 \begin{pmatrix} 0 \\ 0 \\ m \end{pmatrix}$$

$\varphi' = 0 \qquad \sigma = N \qquad |c_g| = 0 \qquad \vec{c}_g \propto \begin{pmatrix} 0 \\ 0 \\ m \end{pmatrix}$ & $\vec{c}_p \propto \begin{pmatrix} k \\ l \\ 0 \end{pmatrix}$

buoyancy oscillations

vertical energy propagation and motion

horizontal ~~motions~~

$\varphi' = 90° \qquad \sigma \to 0 \qquad |c_g| \neq 0 \qquad \vec{c}_g \propto \begin{pmatrix} k \\ l \\ m \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ m \end{pmatrix} = \begin{pmatrix} k \\ l \\ 0 \end{pmatrix}$ & $\vec{c}_p \propto \begin{pmatrix} 0 \\ 0 \\ m \end{pmatrix}$

steady state Taylor-like colums

horizontal energy propagation and motion

vertical ~~particle motions~~

vertical wave number $\qquad \frac{\nabla p}{\rho_0} - \frac{g}{\rho_0} \rho \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0$

ρ does not vary with time

& can be absorbed into ρ_0

& $w = 0$ from density equation

& $u_x + v_y = 0$ from continuity

& each level moves independent without buoyancy effects

$\boxed{\text{Case 3:}}$ $N \neq 0$, $f \neq 0$

$\sigma^2 = N^2 \cos\varphi' + f^2 \sin\varphi'$ $\qquad |c_g| = \underbrace{(N^2 - f^2)}_{\sigma\, \mathbb{R}} \sin\varphi' \cos\varphi'$

$\vec{c}_g \propto (N^2-\sigma^2)\begin{pmatrix} k \\ l \\ m \end{pmatrix} - (N^2-f^2)\begin{pmatrix} 0 \\ 0 \\ m \end{pmatrix}$

$\varphi' = 0$ $\qquad \sigma = N$ $\qquad |c_g| = 0$ $\qquad \vec{c}_g \propto (f^2 - N^2)\begin{pmatrix} 0 \\ 0 \\ m \end{pmatrix}$ & $c_p \propto \begin{pmatrix} k \\ l \\ 0 \end{pmatrix}$

buoyancy oscillations $\qquad$ vertical energy propagation and motion $\qquad$ horizontal ~~motions~~

$\varphi' = 90°$ $\qquad \sigma = f$ $\qquad |c_g| = 0$ $\qquad \vec{c}_g \propto \begin{pmatrix} k \\ l \\ m \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ m \end{pmatrix} = \begin{pmatrix} k \\ l \\ 0 \end{pmatrix}$ & $\vec{c}_p \propto \begin{pmatrix} 0 \\ 0 \\ m \end{pmatrix}$

inertial oscillations $\qquad$ horizontal energy propagation and motion $\qquad$ vertical ~~motions~~

$0 < \sigma < N$

What happens for σ outside the range (f, N) ?

$$w_{zz} - \left(\frac{N^2-\sigma^2}{\sigma^2-f^2}\right)(w_{xx}+w_{yy}) = 0 \qquad N^2 > \sigma^2, \quad \sigma^2 > f^2$$

$\rightarrow$ hyperbolic

$$w_{zz} + \left(\frac{\sigma^2-N^2}{\sigma^2-f^2}\right) w_{xx}+w_{yy} = 0 \qquad N^2 < \sigma^2$$

$\rightarrow$ elliptic

$$m^2 = -\left(\frac{\sigma^2-N^2}{\sigma^2-f^2}\right)(k^2+l^2)$$

& complex, decaying sol.

Wave Guide Modes (surface + bottom BC)

$$w_{zz} - \left(\frac{N^2 - \sigma^2}{\sigma^2 - f^2} \right) \left(w_{xx} + w_{yy} \right) = 0$$

At the free surface $z \approx 0$

hydrostatic

perturbation

$$\frac{Dp^*}{Dt} = 0 \qquad \text{total pressure } p^* = p_0 + p$$

$$\frac{\partial p^*}{\partial t} = \frac{\partial p}{\partial t}$$

$0, \; p_0 = p_0(z)$

$$\frac{\partial p^*}{\partial t} + u \cancel{\frac{\partial p_0}{\partial x}} + v \cancel{\frac{\partial p_0}{\partial y}} + w \frac{\partial p_0}{\partial z} = 0$$

$$\frac{\partial p}{\partial t} + w \frac{\partial p_0}{\partial z} = 0 \qquad \frac{\partial p_0}{\partial z} = -\rho_0 g$$

$$-i\sigma p - w g \rho_0 = 0$$

$\nabla_H^2 \cdot$

$$-i\sigma \nabla_H^2 p - \rho_0 g \nabla_H^2 w = 0$$

Page-50, eq.-4

Text

Recall continuity $\quad -i\sigma \nabla_H^2 p = -(\sigma^2 - f^2) \rho_0 w_z$

$\Rightarrow$

$$-(\sigma^2 - f^2) \cancel{\rho_0} w_z - \cancel{\rho_0} g \nabla_H^2 w = 0$$

and BC is

$$\boxed{(\sigma^2 - f^2) w_z + g \nabla_H^2 w = 0 \quad \text{at } z=0}$$

More general than the rigid lid assumption w = 0 at z=0

Governing Equation:
(same as before)

$$w_{zz} - \underbrace{\frac{(N^2-\sigma^2)}{(\sigma^2-f^2)}}\,(w_{xx}+w_{yy}) = 0$$

Boundary Conditions:
(new)

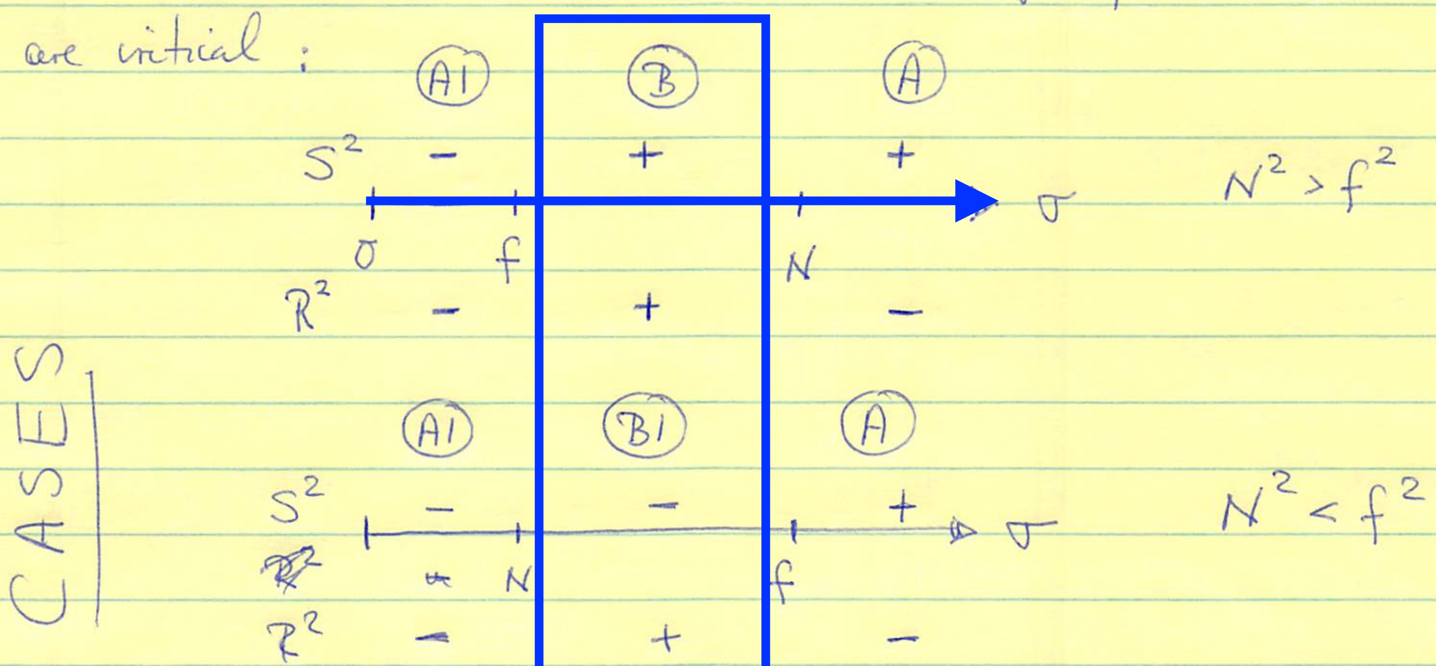
$$w_z + \frac{g}{\underline{(\sigma^2-f^2)}}(w_{xx}+w_{yy}) = 0 \qquad @ \; z \approx 0$$

$$w = 0 \qquad @ \; z = -D$$

Consider $N^2 = \text{const.}$

Sign of $S^2 \equiv \sigma^2 - f^2 \qquad R^2 \equiv \dfrac{N^2-\sigma^2}{\sigma^2-f^2} = \dfrac{N^2-\sigma^2}{S^2}$

are critical:



Wave Solutions exp[i*k*x - i*s*t]:	None	Discrete Set, Modes	One
Decaying Solutions exp[k*x - i*s*t]	Discrete Set	One	Discrete Set

Case A: $R^2 = \dfrac{N^2-\sigma^2}{S^2} < 0$ $w_{zz} + R_1^2 (w_{xx}+w_{yy}) = 0$

$R_1^2 = -R^2 > 0$

Let $w = w(z) \cdot e^{-i\sigma t + ikx}$

then $(\sigma^2 - f^2)\, w_z - g\, k^2\, w = 0$ @ $z = 0$

$w_{zz} + k^2\, w = 0$

$w = 0$ @ $z = -D$

$\downarrow$ $w = e^{-i\sigma t + ikx} \underbrace{\sinh\left(+k R_1 (z+D)\right)}_{w(z)}$

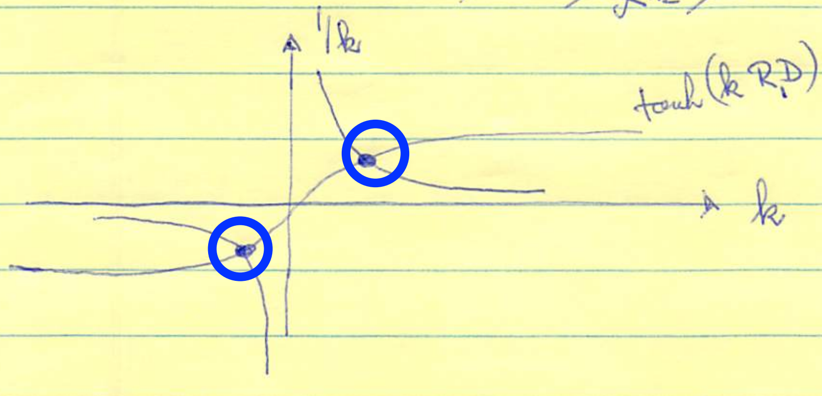
which satisfies the BC @ $z=0$ if and only if

transzendental dispersion $R_1 = R_1(\sigma)$

$$\boxed{+R_1 \frac{(\sigma^2 - f^2)}{g k} = \tanh(+k R_1 D)}$$

$\downarrow$ $\cancel{+R \frac{(\sigma^2-f^2)}{gk} = \tanh(k R D)}$

$\sinh(x) = -\sinh(-x)$
$\cosh(x) = \cosh(-x)$
$\tanh(x) = -\tanh(-x)$



Two waves traveling in opposite directions

[properties similar to $N^2 = f^2 = 0$]

"like" surface gravity waves

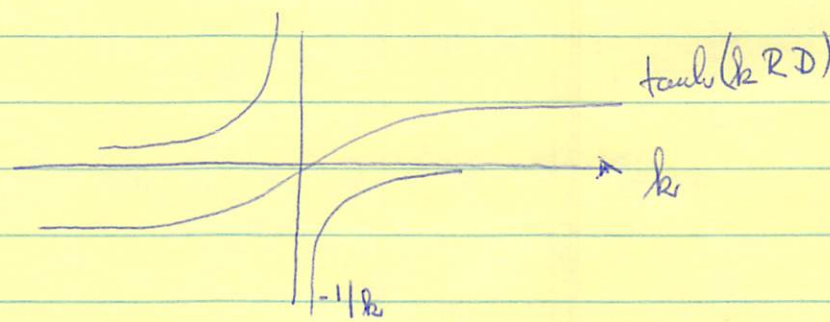
Note that for $S = \sigma^2 - f^2 < 0$ (Case A1)

no waves exist, because

$$+\frac{R_1(\sigma^2-f^2)}{gk} = \tanh kRD$$

becomes

$$-\frac{R_1 S^2}{gk} = \tanh(kRD)$$



Our assumption of free waves in x-direction is impossible to satisfy

Case B : $R^2 > 0$

$w_{zz} - R^2 (w_{xx} + w_{yy}) = 0$

$$w = e^{i\sigma t + ikx}$$

and equations give $(\sigma^2 - f^2)\, w_z - g k^2 w = 0 \quad @\, z=0$

$$w_{zz} + k^2 R^2 w = 0$$

$$w = 0 \qquad @\, z=-D$$

Solutions are $w_{zz} - R^2 (w_{xx} + w_{yy}) = 0$

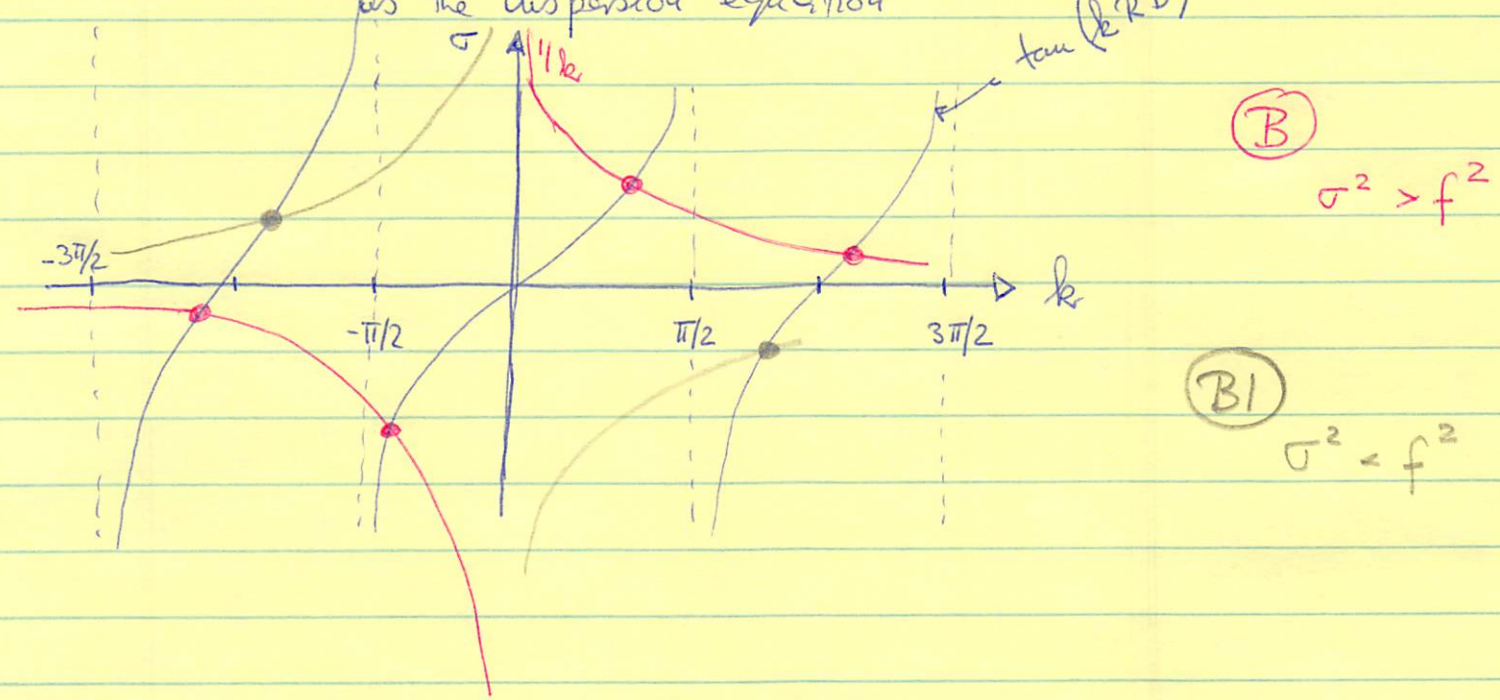
$$w = e^{-i\sigma t + kx} \cdot \sin[kR(z+D)]$$

with

transzendental equation $R = R(\sigma)$

$$\frac{R(\sigma^2 - f^2)}{gk} = \tan(kRD)$$

as the dispersion equation



There is now an infinite set of traveling modes at discrete frequencies of opposite direction $k > 0$ and $k < 0$

goto 70.2

that is, we have a set of $n = 1, 2, 3, \ldots$ "modes"

For large k

$$k_n R D = \pm n \cdot \pi$$

$$k_n \left(\frac{N^2 - \sigma^2}{\sigma^2 - f^2} \right)^{1/2} D = \pm n \cdot \pi$$

1/k —> 0
thus
tan(kRD) —> 0
thus
kRD —> pi

$$\underbrace{k_n D \left(\frac{N^2-\sigma^2}{\sigma^2-f^2}\right)^{1/2}}_{\ll 1} = \pm n\pi$$

$n = 0$

$$k_0^2 D^2 \frac{(N^2-\sigma^2)}{\sigma^2-f^2} = 0$$

$$k_0^2 D^2 N^2 = \sigma^2$$

$$k_0^2 = \frac{\sigma^2}{D^2 N^2} = \frac{\sigma^2}{D^2 \cdot (-) g \partial_z \rho_0} \rho_0$$

$$k_n^2 D^2 \frac{N^2-\sigma^2}{\sigma^2-f^2} = n^2\pi^2$$

$$\frac{\cancel{R}(\sigma^2-f^2)}{g k} = k \cancel{R} D$$

$$\sigma^2 - f^2 = g k^2 D \quad \text{or} \quad k^2 = \frac{\sigma^2-f^2}{(g D)}$$