$$w_z + g/S'^2 \, \nabla_H^2 w = 0 \qquad S^2 \equiv \sigma^2 - f^2$$

$z = 0$

$$\boxed{w_{zz} - R^2 \, \nabla_H^2 w = 0} \qquad R^2 \equiv \frac{N^2 - \sigma^2}{\sigma^2 - f^2}$$

$z = -D$

$w = 0$

Consider $N^2 = \text{const.}$ and different signs of S'^2 and R^2 to reflect different dynamics

	Case-A1 (0 to f)	$\boxed{\text{Case-B}}$ (f to N)	Case-A ($> N$)	
S'^2:	$-$	$+$	$+$	$N^2 > f^2$
R^2:	$-$	$+$	$-$	
	no solution at all	regular set of wave solutions	one solution only	

Wave Frequency σ

	Case-A1 (0 to N)	$\boxed{\text{Case-B1}}$ (N to f)	Case-A ($> f$)	
	— " —	— " —	— " —	
S^2	$-$	$-$	$+$	$N^2 < f^2$
R^2	$-$	$+$	$-$	

σ

Case - B : $R^2 > 0$

Try $\underline{w = w_0 e^{-i\sigma t + ik x}}$ $\qquad \underline{w_0 = w_0}(z)$

(1) @ $z = 0$ $\qquad w_{0z} + (ik)^2 \, g / s'^2 \underline{w_0} = 0$

(2) $\qquad w_{0zz} - (ik)^2 \, R^2 \, w_0 = 0$

(3) @ $z = -D$ $\qquad w_0 = 0$

Try $\underline{w_0(z)} = \sin \underline{[kR\,(z+D)}]$

$$\frac{\partial w_0}{\partial z} = \cos[kR\,(z+D)] \cdot kR$$

$$\frac{\partial^2 w_0}{\partial z^2} = -\sin[kR\,(z+D)] \cdot (kR)^2$$

$\downarrow$

@ $\underline{z=0}$: $kR \cos[kRD] - \dfrac{k^2 g}{s'^2} \underline{\sin[kRD}] = 0$

(1) $\boxed{\downarrow \quad \tan[kRD] = \dfrac{kRS^2}{k^2 g}}$ Dispersion

(2) $(kR)^2 (-) \sin(\ldots) + (kR)^2 \sin(\ldots) = 0$ ✓

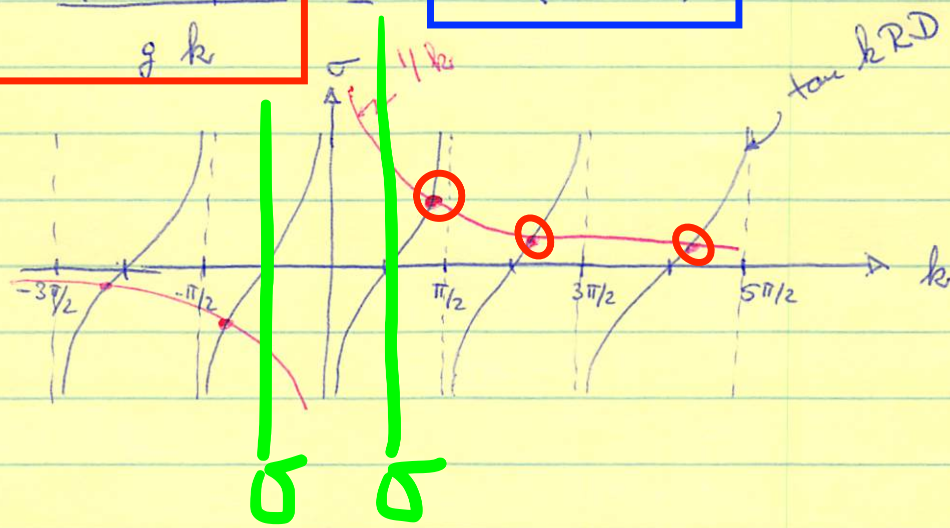
(3) @ $z = -D$: $\sin[kR \cdot 0] = 0$ ✓

Start with Dispersion Relation

Case B: $R^2 \equiv \frac{N^2-\sigma^2}{\sigma^2-f^2} > 0$ and $\sigma^2 > f^2$
($N^2 > \sigma^2$)

Case B1: $\sigma^2 < f^2$
($N^2 < \sigma^2$)

$$\boxed{R\,\frac{(\sigma^2-f^2)}{g\,k}} = \boxed{\tan(k R D)}$$



What's wrong with this plot?

For large k $\quad 1/k \rightarrow 0$ & $\tan(kRD) \rightarrow 0$ also

&

$k_n R D \approx n \cdot \pi$

For $\boxed{k \ll 1}$ $\quad \tan(kDR) \approx kDR$ & $\frac{\cancel{R}(\sigma^2-f^2)}{g\,k} \approx k D \cancel{R}$

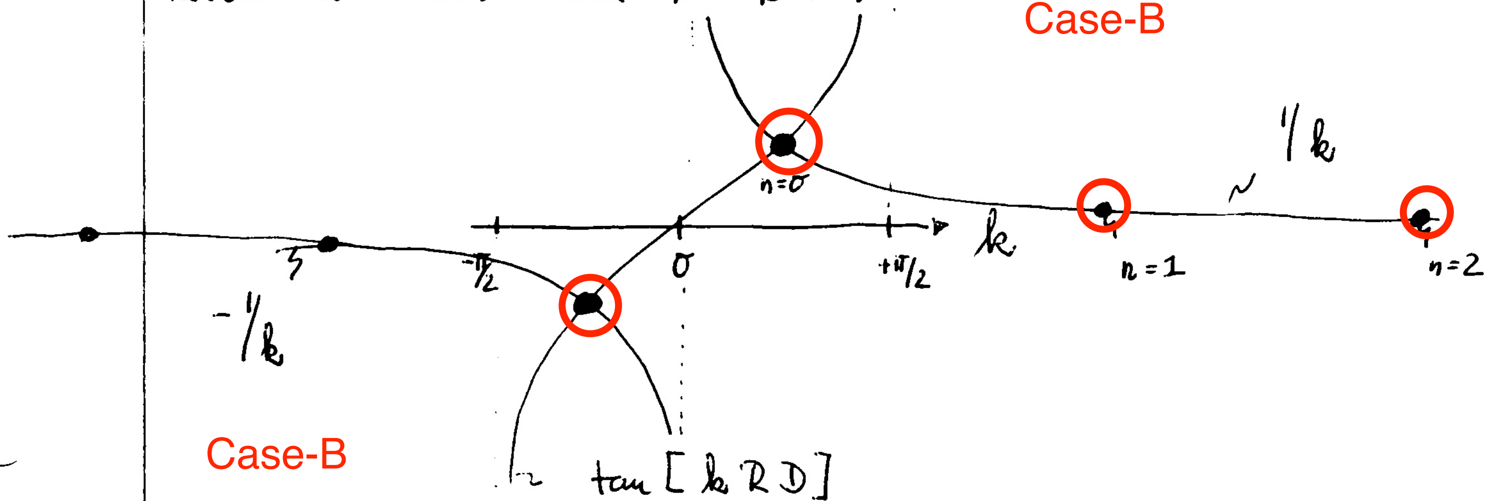
$N = -\frac{g}{\rho_0}\partial_z \rho_0$ & $k_0^2 \approx \frac{\sigma^2-f^2}{g\,D}$

as R drops out, so does any "internal" or "baroclinic" & barotropic mode

Dispersion Relation for Case-B

$$\tan\left[k R D\right] = \frac{R S'^2}{g} \cdot \frac{1}{k}$$

Recall $R = R(\sigma)$ and $S'^2 = S'^2(\sigma)$



For $k \gg 1$ $\quad 1/k \to 0$ $\quad \Rightarrow \tan(kRD) \to 0$ also

(baroclinic modes) $\quad \Rightarrow \boxed{k_n R D \approx n \cdot \pi}$

$n = 1, 2, 3, \ldots$

For $k \ll 1$ $\quad \tan(kRD) \approx k R D \quad \Rightarrow$

(barotropic mode) $\quad \Rightarrow \quad k \cancel{R} D = \frac{\cancel{R} S'^2}{g} \cdot \frac{1}{k}$

$n = 0$

or $\quad g D = S^2 / k^2$

or $\quad \boxed{k^2 = (\sigma^2 - f^2) / g D}$

Baroclinic Modes

$$\boxed{k_n R D \cong n\pi}$$

Dispersion

$$k_n^2 D^2 (N^2-\sigma^2) = n^2\pi^2(\sigma^2-f^2)$$

$$N^2-\sigma^2 = \left(\frac{n\pi}{k_n D}\right)^2(\sigma^2-f^2)$$

$$N^2 + f^2\left(\frac{n\pi}{k_n D}\right)^2 = \left(\frac{n\pi}{k_n D}\right)^2\sigma^2+\sigma^2$$

$$= \sigma^2\left[1+\left(\frac{n\pi}{k_n D}\right)^2\right]$$

$$\kappa^{\cancel{2}} = \frac{n\pi}{k_n D}$$

$$\sigma^2 = \cancel{N^2\left(\frac{\pi n}{k_n D}\right)^2 +} f^2 \quad \frac{N^2+f^2\kappa^2}{1+\kappa^2}$$

$$\boxed{\sigma^2 = \frac{N^2+f^2\kappa^2}{1+\kappa^2}} = \frac{f^2\left(\frac{N^2}{f^2}+\kappa^2\right)}{1+\kappa^2} \qquad \boxed{\kappa = \frac{n\pi}{k_n D}}$$

shallow water $k_n D \ll 1$ or $(k_n D)^{-1} \gg 1$:
$\kappa^2 \gg 1$

$$\boxed{\sigma^2 = \frac{N^2}{\kappa^2}+f^2} = f^2 + N^2\left(\frac{k_n D}{n\pi}\right)^2$$

Barotropic Mode:

With
$$\frac{R\,(\sigma^2-f^2)}{g\,k} = \tan(kRD)$$

for $k_0 RD \ll 1$ $\quad \tan k_0 RD \approx k_0 RD$ and

$$\frac{\cancel{R}\,(\sigma^2-f^2)}{g\,k_0} = k_0 \cancel{R} D$$

as $R = \frac{N^2-\sigma^2}{\sigma^2-f^2}$ drops out, there is no N anywhere & barotropic mode

& $\quad k_0^2 = \frac{\sigma^2-f^2}{gD}$ $\quad$ which is our old surface gravity wave in shallow water modified by rotation f.

Exists only for $\sigma^2 > f^2$

If our surface BC were a rigid lid $(w = 0 \;@\; z = 0)$ then the dispersion relation becomes

$$\sin(kRD) = 0$$

or $\quad k_n RD = \pm n\pi \quad$ is exact?

or

$$k_n^2 \cdot \frac{N^2-\sigma^2}{\sigma^2-f^2} D^2 = n^2 \pi^2$$

For the surface mode we have

$$\sigma^2 \approx f^2 + k^2 g D$$

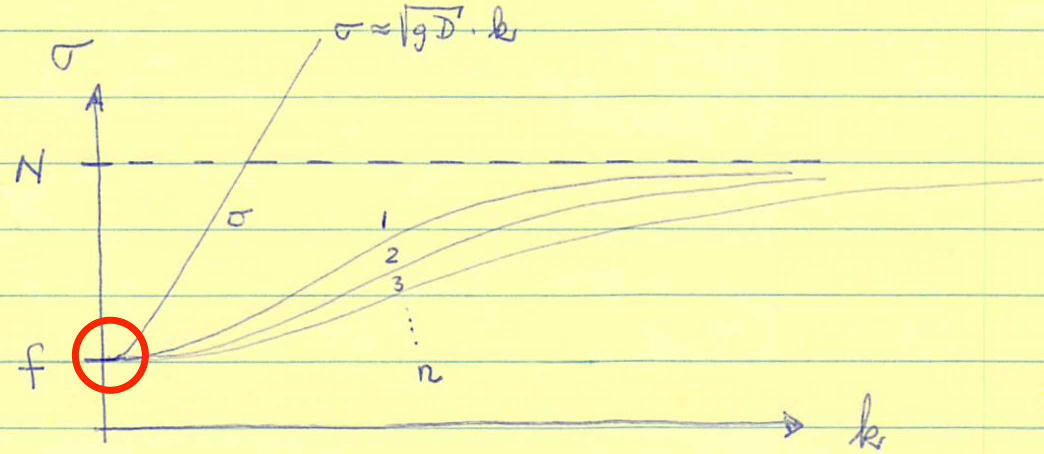
(rigid lid loses this mode)

while for the internal modes we have

$$(\sigma^2 - f^2)\left(\frac{n\pi}{kD}\right)^2 \approx N^2 - \sigma^2$$

(rigid lid makes this exact)

or $\sigma^2\left[1 + (n\pi/kD)^2\right] \approx N^2 + f^2 (n\pi/kD)^2$



$\sigma > f$

$\sigma < N$

$\vec{c}_g = \frac{\partial \sigma}{\partial k} \longrightarrow 0$ @ $\sigma = f$, $\sigma = N$

Wave Kinematics

$$w = w_0\, e^{-i\sigma t + ikx} \sin\left[kR(z+D)\right]$$

From continuity $\quad u_x + w_z = 0$

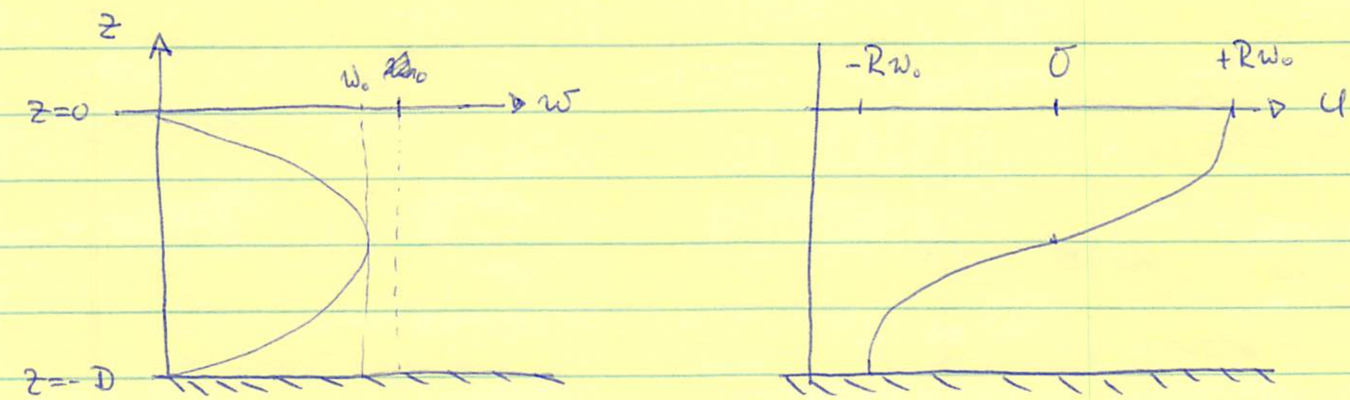
$$u = i\,\underline{R w_0}\, e^{-i\sigma + ikx} \cos\left[kR(z+D)\right]$$

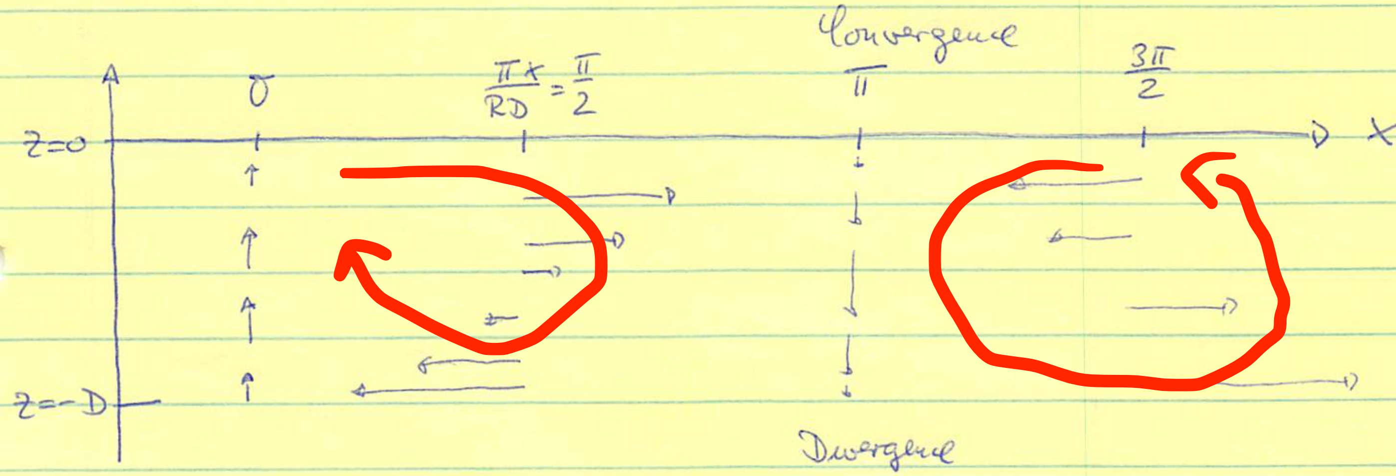
Consider mode $n = 1$

$$k_1 \approx \pi / RD$$

and

$$w = w_0 \cos(k_1 x - \sigma t)\, \sin\left[\pi(z+D)/D\right]$$
$$u = -R w_0 \sin(\quad\ldots\quad)\, \cos\left[\quad\ldots\quad\right]$$





Decaying (Evanescent) Modes

Lets revisit the cases, but lets seek for waves that decay in the x-direction.

Case A: $\boxed{w = e^{-i\sigma t \underline{+ kx}}\, w(z)}$ same BC as before

$\Rightarrow \; w = e^{-i\sigma t + kx} \sin[k R_1 (z+D)]$ $\quad R_1 > 0$

subject to

$$R_1 \frac{(\sigma^2 - f^2)}{g k} = -\tan(k R_1 D) \qquad \text{Dispersion}$$

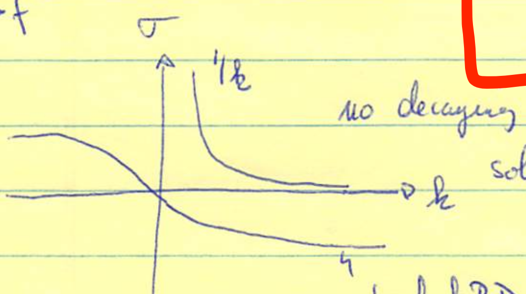
There are an infinite set of modes both for $\sigma^2 > f^2$ (Case A)
and $\sigma^2 < f^2$ (Case A1)

Case B: $w = e^{-i\sigma t + \underline{kx}}\, w(z)$ same BC as before

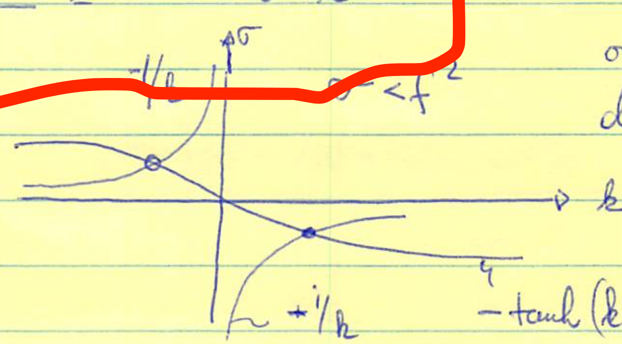
$\Rightarrow \; w = e^{-i\sigma t + kx} \sinh[k R (z+D)]$ $\quad R > 0$

subject to $\boxed{R \dfrac{(\sigma^2 - f^2)}{g k} = -\tanh kRD}$

$\sigma^2 > f^2$



no decaying solutions



only two decaying solutions

Reflection from a Wall or Slope

Consider a 2-D wave $e^{-i\sigma t + ikx + imz}$

that satisfies $w_{zz} - R^2 w_{xx} = 0$ $\qquad R = \frac{N^2 - \sigma^2}{\sigma^2 - f^2}$

$$\boxed{m = \pm R k}$$

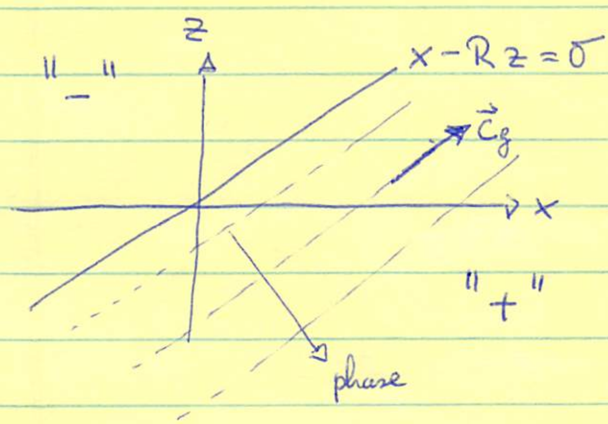
(make c_{gx} and c_{gz} the same magnitude) $c_{gx}/c_{gz} = \pm 1$

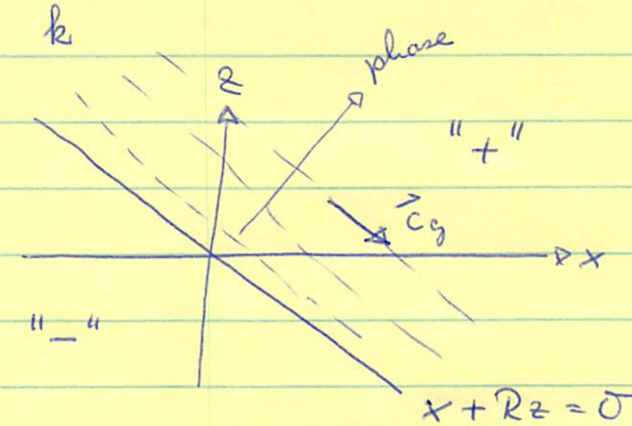
Lines of constant phase are those for whic

$$-\sigma t + kx \pm R k z = \text{const.}$$

or

$$x \pm R z = \frac{\sigma}{k} . t + \text{const}$$



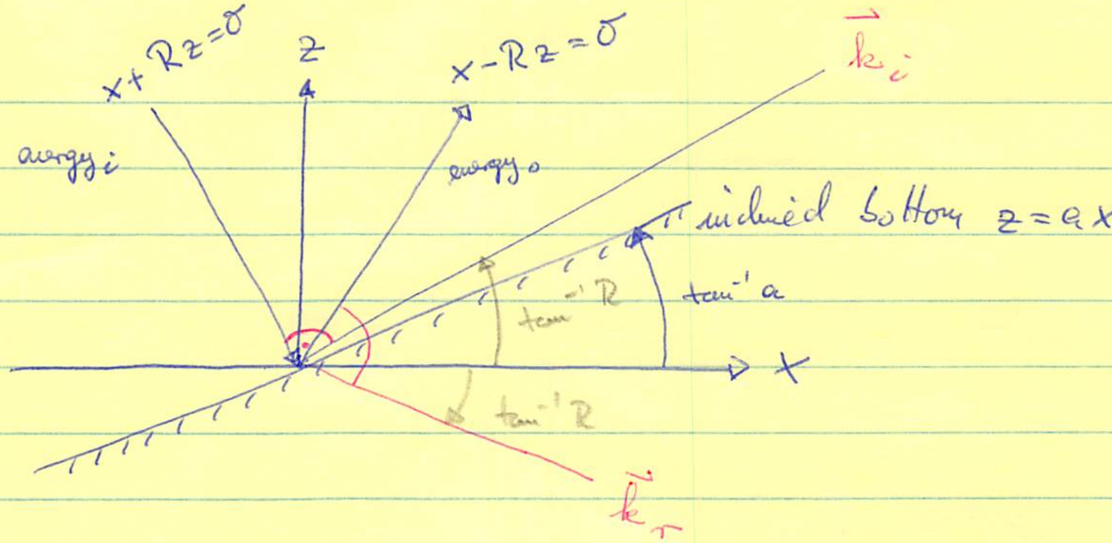


Recall

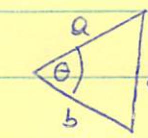
Phase propagates ⊥ to $x \pm Rz = \text{const}$

Energy propagates ∥

Wave numbers $\vec{k}$ can change only via their orientation from the vertical (Dispersion)



~~The normal velocity must vanish on the wall~~



This geometry, law of cosines [$\cos\theta = (a^2+b^2-c^2)/2ac$], and some algebra gives

$$k_r = k_i\left(\frac{1+aR}{1-aR}\right) \qquad m_r = -m_i\left(\frac{1+aR}{1-aR}\right)$$

notice sign *

Internal waves at σ can go only in fixed directions from the vertical φ' ($\pm\tan^{-1}R$ here)
the reflection is <u>NOT</u> normal to the surface, but rather in the direction of the stability vector, that is the vertical

* Notice that there is also the denominator $1-aR$ that can change sign if $aR > 1$ or $a > R$

↳ there is a critical slope $a_c = R$ that backward reflection occurs