## Internal Waves Dispersion $R^2 > \sigma$

$$(\sigma^2 - f^2)\left(\frac{n\pi}{kD}\right)^2 = N^2 - \sigma^2$$

N=const.
k^2 —> k^2+l^2

↓

$$\sigma^2 = \frac{N^2 + f^2 (n\pi/kD)^2}{1 + (n\pi/kD)^2}$$

For shallow water waves $kD \ll 1$ or $\left(\frac{n\pi}{kD}\right)^2 \gg 1$

Depth much smaller than wave length

$$\sigma^2 = \frac{N^2}{(n\pi/kD)^2} + f^2 \cancel{\frac{(n\pi/kD)^2}{(n\pi/kD)^2}}$$

$$\boxed{\sigma^2 = \left(\frac{ND}{n\pi}\right)^2 \cdot \frac{1}{k^{-2}} + f^2}$$

Different Names
Same Thing,
Inertia-Gravity Waves
Poincaré Waves
Sverdrup Waves

$$\underbrace{c_p^2 = \frac{\sigma^2}{k^2} = \underbrace{\left(\frac{ND}{n\pi}\right)^2}_{\equiv g \cdot D_n}}_{\text{non-dispersive for } f=0} + \frac{f^2}{k^2}$$

dispersive $f = 0$

# Shallow Water Gravity Waves

$$\sigma^2 = \left(\frac{N \cdot D}{n\pi}\right)^2 \cdot k^2 + f^2$$

or with

$$D_n \equiv \left(\frac{N D}{n\pi}\right)^2 \cdot \frac{1}{g}$$

$$\sigma^2 = g D_n \cdot k^2 + f \qquad 1\text{-}D$$

$$\underbrace{\sigma^2 = g D_n \left(k^2 + l^2\right) + f}_{\text{dispersive for } f \neq \sigma \quad \text{non-dispersive } f = \sigma} \qquad 2\text{-}D$$

- Wave Reflections from Wall $(f = 0)$
- Seiche Motion in closed basins $(f = 0)$
- Depth Change $(f = \sigma)$ refraction
- Edge Waves $(f = 0)$ trapped modes

- Sverdrup Waves $(f \neq 0)$ $\sigma > f$
- Kelvin Waves $(f \neq 0)$ $\sigma > f$ and $\sigma < f$
- Topographic Rossby Waves $(f \neq 0)$ $\sigma < f$
- Rossby Waves $\left(f \neq 0, \frac{df}{dy} = \beta\right)$ $\sigma < f$

# Shallow Water Equations with Rotation

(1) $u_t - fv = -\frac{1}{\rho_0} p_x$ x-momentum

(2) $v_t + fu = -\frac{1}{\rho_0} p_y$ y-momentum

(3) $0 = -\frac{1}{\rho_0} p_z - \frac{g\rho}{\rho_0}$ z-momentum

no vertical accelerations dw/dt !!!
(compare to page-49)

(4) $\boxed{\rho_t + w\,\rho_{0z} = 0}$ density equation

(5) $u_x + v_y + w_z = 0$ continuity equation

$f = 2\Omega \sin\Theta$ where $\Theta$ is latitude and $\Omega$ is magnitude of earth's rotation

Recall that density $\underline{\rho^* = \rho_0(z) + \rho(x,y,z,t)}$ with $\rho \ll \rho_0$

As before we removed the hydrostatic part and used the Boussinesq Approximation [p. 51] ($\rho_0(z)$ is constant <u>except</u> in density equation)

Combine vertical momentum with density equation to yield

(3') $N^2 w = -\frac{1}{\rho_0} p_{zt}$ $N^2 = \frac{-g}{\rho_0}\frac{\partial \rho_0}{\partial z}$

N=N(z) !!!

Skip 2020

Lets assume homogeneous fluid, that is, $N^2 = 0$

Homogeneous Fluid (Barotropic Motions)

Hydrostatic vertical momentum

$$p_z = -g\rho_0$$ ($p$ is total pressure not pertubation)

integrate

$$p(z=\eta) - p(z) = -g\rho_0(\eta - z)$$

or

$$p(z) = \underbrace{p(z=\eta)}_{p_{\text{atmosphere}}=0} + g\rho(\eta - z)$$

$$= +g\rho(\eta - z)$$

$$u_t - fv = -g\eta_x$$

$$v_t + fu = +g\eta_y$$

$$u_x + v_y + w_z = 0$$

integrate continuity ~~eqns~~ from $z=-D$ to $z=\eta$

$$\int_{-D}^{\eta}(u_x + v_y)\,dz + w\Big|_{z=\eta} - w\Big|_{z=-D} = 0$$

BC are

$w = \dfrac{D\eta}{Dt}$ at $z=\eta$ and $w = -uD_x - vD_y$

at $z=-D$

$$\eta_t + [u(\eta + D)]_x + [v(\eta + D)]_y = 0$$

For $\eta \ll D$ $\quad$ ~~~~ sh $\quad \eta_t + (uD)_x + (vD)_y = 0$

These are the so-called shallow water equations with rotation.
(non-rotating version was derived in surface gravity waves)

Skip 2020

Lets include stratification in constant depth ($D$ = const) ocean

$u = U(x,y,t) \cdot F(z)$

$v = V(x,y,t) \cdot F(z)$

$w = W(x,y,t) \cdot G(z)$

$p = P(x,y,t) \cdot H(z)$

Method: Separation of variables

(1) $\quad (U_t - fV) \cdot F = -\frac{1}{\rho_0} P_x H \qquad\qquad u_t - fv = -\frac{1}{\rho_0} p_x$

(2) $\quad (V_t + fU) \cdot F = -\frac{1}{\rho_0} P_y H \qquad\qquad v_t + fu = -\frac{1}{\rho_0} p_y$

(3) $\quad N^2 W G = -\frac{1}{\rho_0} P_t H_z \qquad\qquad N^2 w = -\frac{1}{\rho_0} p_{zt}$

(4) $\quad (U_x + V_y) F + W G_z = 0 \qquad\qquad u_x + v_y + w_z = 0$

$$u = U \cdot F(z)$$
$$v = V \cdot F(z)$$
$$w = W \cdot G(z)$$
$$p = P \cdot H(z)$$

with $H = g\rho_0 F$

(1) $U_t - fV = -\frac{1}{\rho_0} P_x \frac{H}{F} \longrightarrow U_t - fV = -g P_x$

(2) $V_t + fU = -\frac{1}{\rho_0} P_y \frac{H}{F} \longrightarrow V_t + fU = -g P_y$

(3) $N^2 WG = -\frac{1}{\rho_0} P_t H_z \longrightarrow \boxed{N^2 WG = -g P_t F_z}$

For vertical momentum and continuity we require now that

$G_z = F / D_n$ or $\boxed{F = D_n \cdot G_z}$ where $D_n$ is an unknown scale depth

(3) $N^2 WG = -g P_t F_z \longrightarrow \boxed{N^2 WG = -g P_t G_{zz} \cdot D_n}$

(4) $(U_x + V_x) \cdot F + W G_z = 0 \longrightarrow (U_x + V_y) \cdot D_n \cdot \cancel{G_z} + W \cancel{G_z} = 0$

And finally with $W = P_t$ (think of $P$ as the height of an isopycnal, then it time derivative may become a vertical velocity

(3) $N^2 WG = -g P_t G_{zz} \longrightarrow N^2 G = -g D_n G_{zz}$

(4) $(U_x + V_y) \cdot D_n + W = 0 \longrightarrow P_t = -(U_x + V_y) \cdot D_n$

To summarize

(1) $$U_t - fV = -g\,P_x$$

(2) $$V_t + fU = -g\,P_y$$

(3) $$P_t + D_n(U_x + V_y) = 0$$

(4) $$\boxed{G_{zz} + \frac{N^2}{g D_n} G = 0}$$

(5) $G_z - \dfrac{G}{D_n} = 0 \qquad z = 0$

(6) $G = 0 \qquad z = -D$

[See below]

Recall that we do not yet know what $D_n$ is which "scales" the vertical variation of $G$ ($w = W \cdot G(z)$) relative to those ~~of~~ in the horizontal $((u,v) = (U,V)\cdot F(z))$ via

$$F = D_n \cdot G_z$$

## Need Boundary conditions

(5) $$p = g\rho_0 \eta \qquad \text{at} \quad z = 0$$

$$\Rightarrow \quad p_t = g\rho_0 \eta_t \quad \Rightarrow \quad p_t = g\rho_0 w$$

Or in separated variables

$$H \cdot P_t = g\rho_0 W G \quad \text{or} \quad H_z \cdot P_t = g\rho_0 W G_z \quad @\, z=0$$

$$H = g\rho_0 G \qquad [W = P_t]$$

$$g\rho_0 F = g\rho_0 G \qquad [H = g\rho_0 F]$$

$$\Rightarrow \quad D_n G_z = G \qquad [F = D_n \cdot G_z] \qquad \text{at } z = 0$$

$$\boxed{G_{zz} + \frac{N^2(z)}{g D_n} G = 0}$$

$$\boxed{N^2 = N^2(z)}$$

$$G_z - \frac{1}{D_n} G = 0 \qquad z = 0$$

$$G = 0 \qquad z = -D$$

This describes an eigen value problem for

$$G(z) = \sin\left[\frac{n\pi}{D}(z+D)\right]$$

where the eigen values $D_n$ are defined such that

$$\frac{N^2}{g D_n} = \frac{\pi^2 n^2}{D^2}$$

or

$$\boxed{D_n = \frac{N^2 D^2}{g n^2 \pi^2}}$$

such that

$$G_{zz} + \underbrace{\frac{n^2\pi^2}{D^2}}\,. G = 0$$

This reduces the above coefficient $N^2(z)/g D_n$ to a constant!

We thus have an infinite set of "modal" solutions which each have an effective depth $D_n = N^2 D^2 / g n^2 \pi^2$

# Full Modal Shallow Water Equations

We thus have a full set of equations for each mode number $n = 1, 2, 3, \dots$

(1) $U^n_t - f V^n = -g P^n_x$

(2) $V^n_t + f U^n = -g P^n_y$

(3) $P^n_t + D_n ( U^n_x + V^n_y ) = 0$

$$\boxed{D_n = \frac{N^2 D^2}{g n^2 \pi^2}}$$

$$G_n(z) = \sin\left[ \frac{n\pi}{D} (z+h) \right]$$

$$F_n(z) = \cos\left[ \frac{n\pi}{D} (z+h) \right]$$

Recall that we have no $n = 0$ mode (the barotropic case $N = 0$) because we assumed a rigid lid at $z = 0$ with $w = 0$ there

How to interpret $D_n$ the so-called "effective depth?"

Recall that for shallow water the phase speed $C_p = \sqrt{gD}$

For internal waves we had (p. 72)

$$\sigma^2 = \frac{N^2}{1+(n\pi/kD)^2} + \frac{f^2 (n\pi/kD)^2}{[1+(n\pi/kD)^2}$$

shallow water $kD \ll 1$ gives

$$\sigma^2 = \frac{N^2}{(n\pi / kD)^2} + f^2$$

$$\& \quad C_p^2 = \frac{\sigma^2}{k^2} = \underbrace{\frac{N^2}{n^2\pi^2} \cdot D^2}_{= g\, D_n} + \frac{f^2}{k^2}$$

This result was derived for $N^2 = \text{const.}$, but we now derived a similar result for $N^2 = N^2(z)$

So the eigenvalue $D_n$ is the effective depth for an internal wave of mode "n" that plays the same role as the total depth does for the barotropic mode

Note that the modes are uncoupled, that is, each mode is independent and can be solved the same way that the barotropic shallow water equation with the ONLY modification that

$$D \longrightarrow D_n = N^2 D^2 / g\, n^2 \pi^2$$

Hence mode-1 has $c_p = \sqrt{g\, D_n} = \frac{N D}{n\pi} = \frac{N \cdot D}{\pi}$

mode-2 has $c_p = \sqrt{g\, D_n} = \frac{N \cdot D}{2\pi}$