Skip
2020

5.3 (p107) Reflection from solid wall (Shallow water waves; $f=0$)

$$u_t = -g\eta_x \qquad v_t = -g\eta_y \qquad \eta_t + D(u_x + v_y) = 0$$

Free wave solutions $\eta = e^{-i\sigma t + ikx + ily}$ give

$$u = \frac{gk}{\sigma}\eta \qquad v = \frac{gl}{\sigma}\eta$$

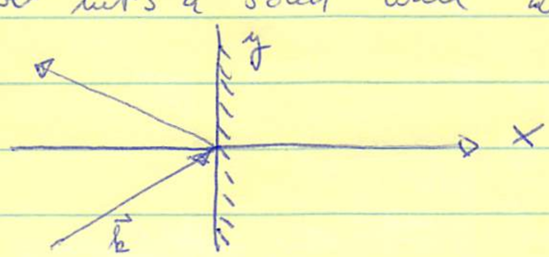
and substitution into continuity gives

$$\sigma^2 = gD(k^2+l^2) = gD\kappa^2 \qquad \kappa^2 = (k^2+l^2)$$

surface gravity waves in shallow water, non-dispersive with

$$c_p = \frac{\sigma}{\kappa} = \sqrt{gD} \qquad |\vec{c}_g| = \frac{d\sigma}{d|\vec{k}|} = \sqrt{gD}$$

Suppose this wave hits a solid wall at $x=0$

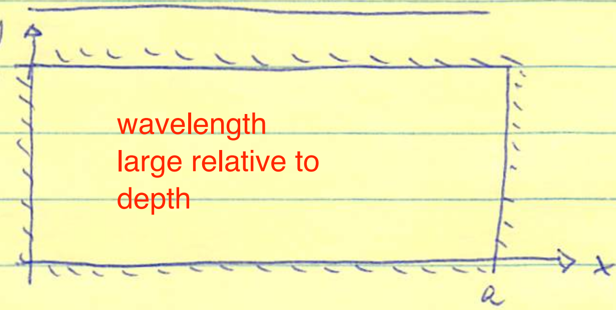


Velocity normal to surface must vanish $u=0$ at $x=0$
and the solution is constructed by adding a wave of same amplitude without phase change

$$\eta = a\,e^{-i\sigma t + ikx + ily} + a\,e^{-i\sigma t - ikx - ily}$$

and the angle of reflection is $\alpha = \tan^{-1}(l/k)$

5.4 (p. 108) **Seiches in a Box**



Think of this as a
- Harbor Basin, or
- Lake, or
- Swimming-pool on ice-breaker

time-dependence $e^{-i\sigma t}$

f=0, slightly modified physics hold also for f=const.

(1) $-i\sigma u = -g\eta_x$ (2) $-i\sigma v = -g\eta_y$ (3) $i\sigma\eta + D(u_x + v_y) = 0$

Use (1) and (2) in (3) to give

$$\boxed{\nabla^2 \eta + \frac{\sigma}{gD}\eta = 0}$$

normal velocities vanish along $x = 0$ and $x = a$ and $y = 0$ and $y = b$

$\eta_x = 0$ at $x = 0, a$ $\eta_y = 0$ at $y = 0, b$

Solutions then are

$$\eta = \cos\left(\frac{n\pi x}{a}\right) \cdot \cos\left(\frac{m\pi y}{b}\right) \qquad n, m = 0, 1, 2, \ldots$$

Which if substituted into the wave equation $\nabla^2\eta + \frac{\sigma}{gD}\eta = 0$

give the dispersion relation

$$\boxed{\sigma^2 = gD\left(\frac{n^2}{a^2} + \frac{m^2}{b^2}\right)\pi^2}$$

These are sometimes refered to as "Eigen-Frequencies," or "Resonance Frequencies"

These are "normal" free modes of a non-rotating ocean.

The gravest modes are $m = 0 \quad n = 1$

$$y = \cos(\pi x / a)$$

has the lowest frequency

$$\sigma^2 = gD\,\pi^2/a^2$$

or $$\boxed{T = \frac{2\pi}{\sigma} = 2a/\sqrt{gD}}$$

which is the time for the wave to cross the basin and go back again. It has a modal line at $x = 2$.

The Baltic Sea, the Black Sea, or Persian Gulf, etc. can all be considered for such "seiche" motions

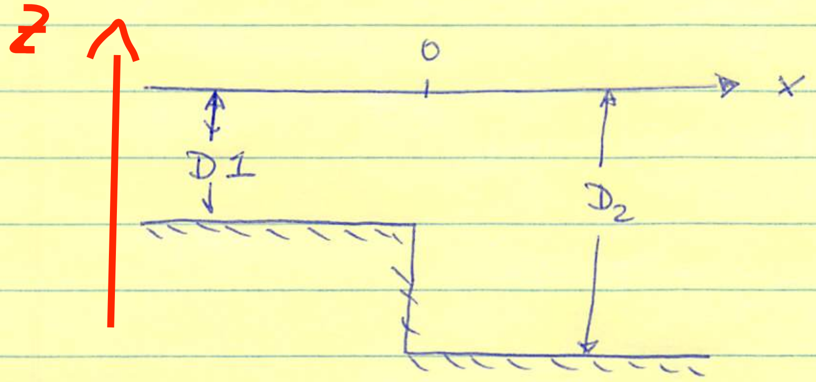
$\longrightarrow$

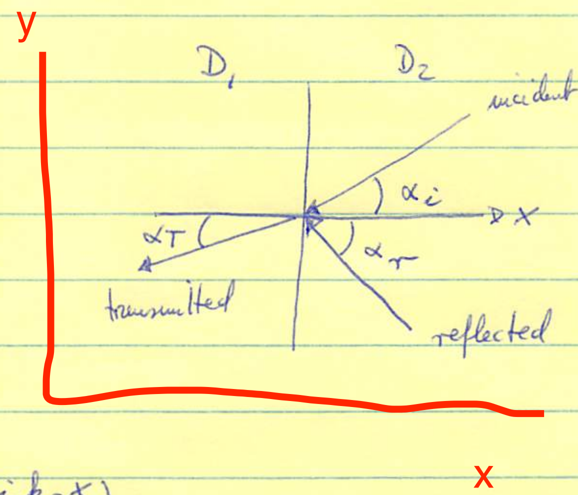
Example:

Baltic Sea —> Krauss (1979)
"Modal" decomposition:
Discrete set of spatial structures
Discrete set of eigen-frequencies

Variable Topography
f=0

5.5 (p. 110) Wave Propagation over a step





$$\eta = e^{-i\sigma t + ily} * A_T e^{-ik_1 x} + e^{-i\sigma t + ily} * \left(A_i e^{-ik_2 x} + A_r e^{+ik_2 x}\right)$$

where the amplitudes A_i, A_r, and A_t are unknown

"Matching Conditions" for both η and $u \cdot D$
on each side of step

Thus at $x=0$ $\quad A_i + A_r = A_t$

$$D_2 k_2 (-A_i + A_r) = D_1 k_1 (-A_t)$$

Treat as 2 equations for the two unknowns A_T and A_R to get

$$A_T = 2 A_i \Big/ \left(1 + \frac{D_1 k_1}{D_2 k_2}\right)$$

incoming: -D2*k2*Ai
reflected: D2*k2*Ar
transmitted: D1*k1*At
incoming=reflected+transmitted

$$A_R = A_i \left(1 - \frac{D_1 k_1}{D_2 k_2}\right) \Big/ \left(1 + \frac{D_1 k_1}{D_2 k_2}\right)$$

Discuss

$D_1 \to 0$ $\qquad A_R = A_i$ ok

$A_T = 2A_i$? not ok, because shallow side does not vanish until it is 0

$D_1 = D_2$ $\qquad A_T = A_i$ ok

$A_R = 0$ ok

Define wavenumbers as

$$K_i = K_r = (k_2^2 + l^2)^{1/2} = \boxed{\sigma / \sqrt{g D_2}}$$

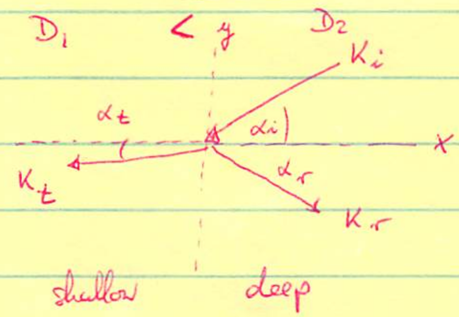
$$K_t = (k_1^2 + l^2)^{1/2} = \boxed{\sigma / \sqrt{g D_1}}$$

Then

$$l = K_i \sin \alpha_i = K_r \sin \alpha_r \quad \text{or} \quad \boxed{\alpha_i = \alpha_r}$$

And

$$l = K_i \sin \alpha_i = K_t \sin \alpha_t$$

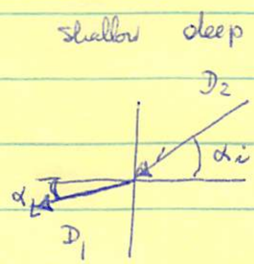


$$\Rightarrow \cancel{\sigma} \frac{\sin \alpha_i}{\sqrt{g D_2}} = \cancel{\sigma} \frac{\sin \alpha_t}{\sqrt{g D_1}}$$

$$\Rightarrow \boxed{\frac{\sin \alpha_i}{c_i} = \frac{\sin \alpha_t}{c_t}}$$

Snell's Law

we discussed this already for acoustic waves on p. 17



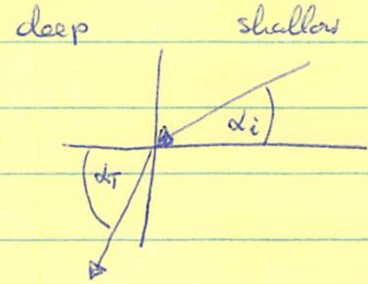
$D_2 > D_1 \Rightarrow c_i > c_t$

$\Rightarrow \alpha_i > \alpha_t$

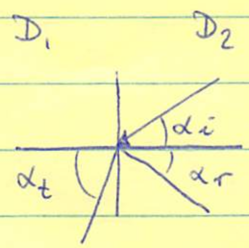
Next imagine the wave comes from shallow to deep

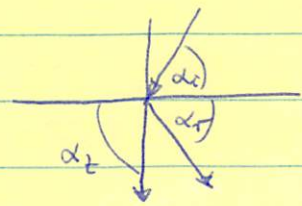
$D_2 < D_1 \;\Rightarrow\; c_i < c_t$

$\Rightarrow\; \alpha_i < \alpha_t$



There exists a critical (trapping) angle at which the wave from shallow cannot transmit into deep water when $\alpha_t = 90°$





$\alpha_i < \alpha_i^c$ $\qquad$ $\boxed{\alpha_i = \alpha_i^c}$

Text

For $\alpha_i = \alpha_i^c$:

$$l = K_i \sin\alpha_i = \underbrace{\frac{\sigma}{\sqrt{gD_2}}}_{K_i} \cdot \underbrace{\sqrt{\frac{D_2}{D_1}}}_{\sin(\alpha_i)} = \frac{\sigma}{\sqrt{gD_1}} = K_t$$

Kt=sqrt(k1^2+l^2)

so $\underline{k_1 = 0}$

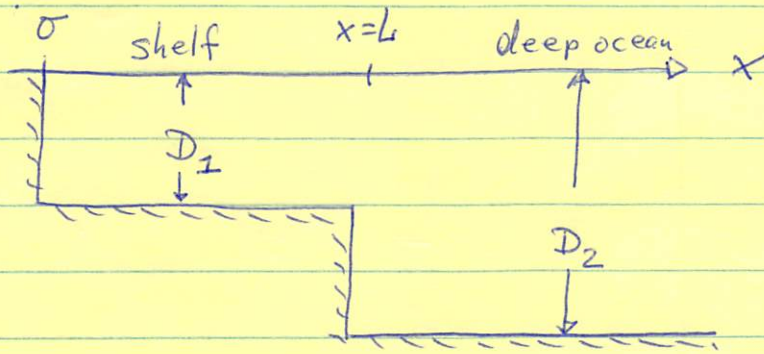
For $\alpha_i > \alpha_i^c$ $\qquad$ $l > K_T = (k_1^2 + l^2)^{1/2} \;\Rightarrow\; k_1 < 0$

$\Rightarrow$ exponential decay away from step

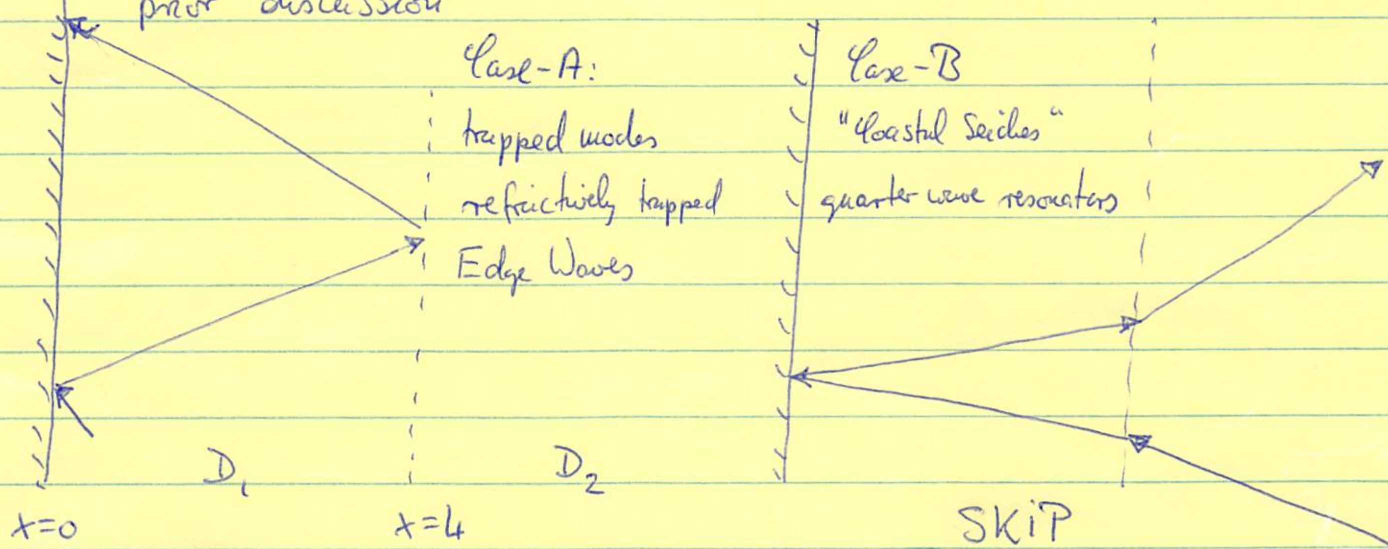
5.6 (p.114)

Edge Waves

Notice that we do not specify a horizontal scale L or vertical scale D1 or D2; they could be ~ 1 m or ~ 1 km



We can expect two type of scenarios based on the prior discussion



SKIP

In each region the elevation satisfies $\nabla^2 \eta + \left(\frac{\sigma^2}{gD}\right)\eta = 0$

Case-A

$\eta = A \cos k_1 x \; e^{ily} \quad 0 < x < L$ trapped wave

$\eta = B e^{-k_2 (x-L)} e^{ily} \quad x > L$ decaying wave

that satisfy $u = 0$ at $x = 0$ and assume internal reflection at shelf edge

$$k_1^2 = \frac{\sigma^2}{gD_1} - \ell^2 \qquad\qquad k_2^2 = \ell^2 - \sigma^2/gD_2$$

see page 94A for details

Both k_1 and k_2 are real for

$$\frac{\sigma^2}{gD_1} > \ell^2 > \frac{\sigma^2}{gD_2} \qquad \text{or} \quad D_1 < D_2$$

Matching η and $\underbrace{u \cdot D}_{D \cdot \partial_x \eta}$ at $x = L$ gives

η : $\qquad A \cos k_1 L = B$

$\partial_x \eta \cdot D$ $\qquad -A D_1 k_1 \sin k_1 L = -D_2 k_2 B$

$\Rightarrow$ Dispersion Eq. $\quad$ or $\quad \tan k_1 L = \dfrac{k_2 D_2}{k_1 D_1} \quad \hat{=} \quad \sigma = \sigma(\ell)$

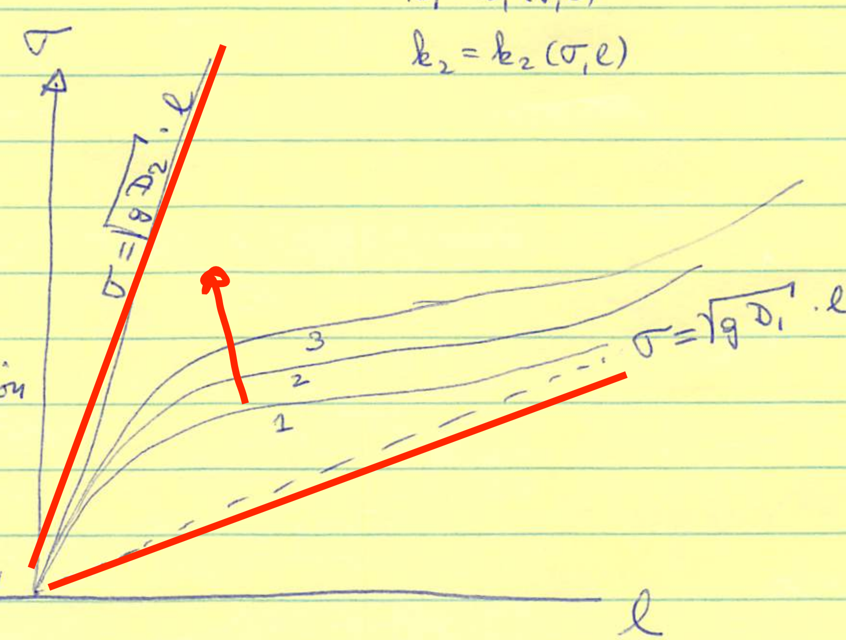
$k_1 = k_1(\sigma, \ell)$
$k_2 = k_2(\sigma, \ell)$

infinite discrete set between
$\sigma = \sqrt{gD_1}\,\ell$
$\sigma = \sqrt{gD_2}\,\ell$

each mode has its own and distinct dispersion relation

For large ℓ

$k_1 \approx (n\pi + \pi/2)/L$



Details on the matching conditions at x=L:

$$\nabla^2 \eta + \frac{\sigma^2}{gD}\eta = 0$$

$$\eta = A\cos k_1 x \; e^{ily} \quad D_1 \qquad\qquad \eta = B e^{-k_2(x-L)} e^{ily} \quad x > L, D_2$$

$$\eta' = -A \sin k_1 x \cdot k_1 \qquad\qquad \eta' = B e^{-k_2(x-L)} \cdot (-) k_2$$

$$\eta'' = -A \cos k_1 x \cdot k_1^2 \qquad\qquad \eta'' = B e^{-k_2(x-L)} (+) k_2$$

$$-k_1^2 + \frac{\sigma^2}{gD_1} - l^2 = 0 \qquad\qquad +k_2^2 + \frac{\sigma^2}{gD_2} - l^2 = 0$$

$$k_1^2 = \frac{\sigma^2}{gD_1} - l^2 \qquad\qquad k_2^2 = l^2 - \frac{\sigma^2}{gD_2}$$

η $\qquad A \cos k_1 L = B \qquad\qquad$ A and B units of η

η_x $\qquad -D_1 k_1 A \sin k_1 L = - D_2 k_2 B$

$$\frac{-D_1 k_1 \cancel{A} \sin k_1 L}{\cancel{A} \cos k_1 L} = - \frac{D_2 k_2 \cancel{B}}{\cancel{B}}$$

$\frac{1}{\cancel{A}}$ $\boxed{\tan k_1 L = + \frac{D_2 k_2}{D_1 k_1}} \qquad\qquad k_1 = k_1(\sigma, l) = \frac{\sigma^2}{gD_1} - l^2$

$$k_2 = k_2(\sigma, l) = l^2 - \frac{\sigma^2}{gD_2}$$

SKIP

Case - B

$$\eta = A \cos k_1 x \qquad 0 < x < L$$

$$\eta = B e^{i k_2 (x-L)} + C e^{-i k_2 (x-L)} \qquad x > L$$

which satisfies $u = 0$ @ $x = 0$ and allows incident (C) and reflected (B) deep ocean waves

$$k_1^2 = \frac{\sigma^2}{g D_1} - l^2 \qquad k_2^2 = \frac{\sigma^2}{g D_2} - l^2$$

$$l < \sigma^2 / g D_2$$

Matching @ $x = L$ of η and $u \partial_x \eta$

η $$A \cos k_1 L = B + C$$

$\eta_x \cdot D$ $$-D_1 k_1 A \sin k_1 L = i D_2 k_2 (B - C)$$

$$A = C \frac{i \cdot 2 \cdot D_2 \cdot k_2}{i D_2 k_2 \cos(k_1 L) - D_1 k_1 \sin(k_1 L)}$$

Wave amplitudes do not drop out as for the (trapped) edge waves

~~For $l = \sigma$ $k_1^2 = \sigma^2 / g D_1$ and $k_2^2 = \sigma^2 / g D_2$ and~~

$$\cancel{\left|\frac{A}{C}\right| = \frac{2 (D_2/g)^{1/2}}{\left[\frac{D_2}{g} \cos^2(k_1 L) + \frac{D_1}{g} \sin^2(k_1 L)\right]^{1/2}}}$$

5.7 Sverdrup and Poincaré Waves

(p.118)

with rotation $f = $ const. time-dependence $e^{-i\sigma t}$

$$u = \frac{g}{\sigma^2 - f^2}\left(f\eta_y - i\sigma\eta_x\right)$$

$$v = \frac{-g}{\sigma^2 - f^2}\left(f\eta_x + i\sigma\eta_y\right)$$

$$\nabla^2\eta + \frac{\sigma^2 - f^2}{gD}\cdot\eta = 0$$

$\eta = e^{ikx + ily}$ gives dispersion

$$\boxed{\sigma^2 = gD\,(k^2 + l^2) + f^2}$$

$\sigma > f$! rotation makes these waves dispersive.

These are long gravity ~~modes~~ waves modified by rotation → Sverdrup Waves

Change axes that $l = 0$ $k = |\vec{k}|$

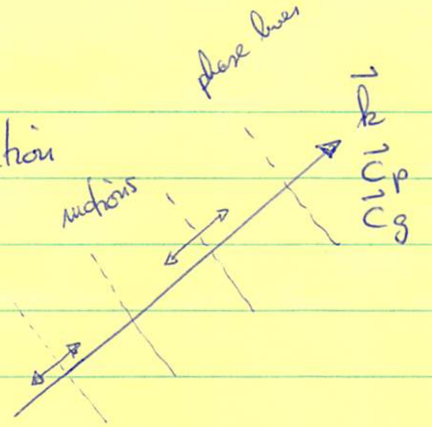
$$\Rightarrow u = \frac{g\eta}{\sigma^2 - f^2}(\sigma k)$$

$$v = \frac{g\eta}{\sigma^2 - f^2}(-ifk)$$

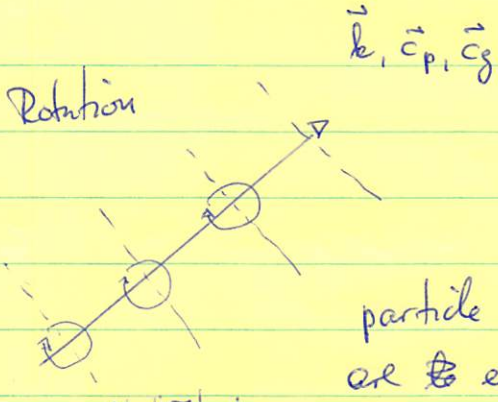
$\Rightarrow \dfrac{u}{v} = \dfrac{i\sigma}{f}$ current ellipse rotating clockwise

require $\sigma \geq f$ low frequency limit

No Rotation



Rotation



$\left|\frac{u}{v}\right| = \left|\frac{i\sigma}{f}\right| < 1$

particle motions are ~~a~~ ellipse with major axis along $\vec{k}$

Rotation makes these waves dispersive

Skip 2020

$$c_{gx} = \frac{gD}{\sigma} k \qquad c_{gy} = \frac{gD}{\sigma} \ell$$

already discussed these waves during internal wave discussions p. 85/86

$N = N(z)$

$$\sigma^2 = \frac{N^2}{(n\pi / kD)^2} + f^2 \qquad \text{internal waves}$$

$$\sigma^2 = gD\,(k^2 + \ell^2) + f^2$$

$$\sigma^2 = gD_n\, k^2 + f^2$$

$$\sigma^2 = \boxed{\frac{N^2 D^2}{(n\pi)^2}} \cdot k^2 + f^2$$

$$D_n = \frac{N^2 D^2}{n^2 \pi^2} \cdot \frac{1}{g}$$

Use same dispersion curve graph to discuss

$$\sigma^2 = gD\,k^2 + f^2$$

$$\sigma = \left(gD\,k^2 + f^2\right)^{1/2}$$

$$\frac{\partial \sigma}{\partial k} = \frac{1}{\cancel{2}}\left(gD\,k^2 + f^2\right)^{-1/2} \cdot \cancel{2}\,gD \cdot k$$

$$= gD \cdot k \Big/ \left(gD\,k^2 + f^2\right)^{1/2}$$

$$c_{gx} = \frac{gD \cdot k}{\sigma} =$$

$$\vec{c}_p = \frac{\sigma}{(k^2+l^2)} \cdot \vec{k} \qquad c_{px} = \frac{\sigma}{(k^2+l^2)} \cdot k = \frac{\left(gD\,k^2 + f^2\right)^{1/2}}{k^2+l^2} \cdot k$$

$$c_{gx} = gD \cdot \frac{k}{\sigma} \cdot \frac{\cancel{k}}{\cancel{k}} \cdot \frac{k^2+l^2}{k^2+l^2} \cdot \frac{\cancel{k}}{k} = \frac{gD\,k^2}{k^2+l^2} \cdot \frac{1}{c_{px}}$$

$$c_{px} = \frac{\sigma \cdot k}{k^2+l^2}$$

$$c_{gx} = \frac{gD\,k^2}{k^2+l^2} \cdot \frac{1}{c_{px}}$$

$$\boxed{\sigma^2 - f^2 = gD\left(k^2 + l^2\right)}$$

If $\sigma \ll f$ → very low frequency $\frac{\partial}{\partial t} \to 0$

steady state

$$-fv = -g\eta_x$$
$$fu = -g\eta_y$$

geostrophic motion

If $\sigma = f$ → $\sigma^2 - f^2 = 0$ → $k=0$ and $l=0$

$$u_t - fv = 0$$
$$v_t + fu = 0$$

$u = \cos\sigma t = \cos ft$

$v = \sin\sigma t = \sin ft$

inertial oscillation in circles

<u>Reflection of Sverdrup Wave from Wall</u> (Poincare Waves)

SKIP

~~Continuity equation~~ $\cancel{-i\sigma\eta_x + f\eta_y = 0 \text{ at } x=0}$

$$u = \frac{g}{\sigma^2 - f^2}\left(f\eta_y - i\sigma\eta_x\right)$$ (p. 96)

$u=0$ at $x=0$

→

$$-i\sigma\eta_x + f\eta_y = 0 \quad \text{at } x=0$$

$\eta = a_i e^{ikx+ily} + a_r e^{-ikx+ily}$ satisfies boundary condition if

$$i\sigma(ika_i - ika_r) + f(ila_i + ila_r) = 0$$

$$\Rightarrow \quad a_r = a_i \, \frac{\sigma k - i f l}{\sigma k + i f l}$$

$f = 0 \;\Rightarrow\; a_r = a_i$ same amplitude, no phase shift

$f \neq 0$ angle of incidence $\tan^{-1}(l/k)$ still equal to angle of reflection,

BUT amplitude differ

$\Rightarrow$ phase shift upon reflection

standing waves normal to wall

reflected waves with phase shift of waves that all travel along the wall.

As long as $\sigma > f$ they exist at any frequency and wave number

$$\sigma^2 = f^2 + gD\,(k^2 + l^2)$$