5.7 Sverdrup and Poincaré Waves

(p.118)

with rotation $f = \text{const.}$ time-dependence $e^{-i\sigma t}$

$$u = \frac{g}{\sigma^2 - f^2}\left(f\eta_y - i\sigma\eta_x\right)$$

$$v = \frac{-g}{\sigma^2 - f^2}\left(f\eta_x + i\sigma\eta_y\right)$$

solve (x,y) momentum equations as two equations for two unkown (u,v)

(see page 96A and 96B)

$$\nabla^2\eta + \frac{\sigma^2 - f^2}{gD}\cdot\eta = 0$$

$\eta = e^{ikx + ily}$ gives dispersion

$$\boxed{\sigma^2 = gD\,(k^2 + l^2) + f^2}$$

$\sigma > f$?

rotation makes these waves dispersive.

These are long gravity ~~modes~~ waves modified by rotation

→ Sverdrup Waves

Change axes that $l = 0$ $k = |\vec{k}|$

$$\Rightarrow u = \frac{g\eta}{\sigma^2 - f^2}\left(\sigma k\right)$$

$$v = \frac{g\eta}{\sigma^2 - f^2}\left(-ifk\right)$$

$$\Rightarrow \boxed{\frac{u}{v} = \frac{i\sigma}{f}}$$

current ellipse rotating clockwise

require $\sigma \geq f$

low frequency limit

Solve (x,y) momentum as 2 equations for 2 unknown (u,v):

$$-i\sigma u \quad - f v \quad = -g\eta_x \qquad |: -i\sigma$$

$$-i\sigma v \quad + f u \quad = -g\eta_y \qquad |: +f$$

$$u - \frac{fv}{-i\sigma} = -\frac{g}{i\sigma}\eta_x$$

$$u - \frac{i\sigma}{f} v = -\frac{g}{f}\eta_y$$

$-$

$$\sigma + \left(+\frac{fi}{\sigma} - \frac{\sigma i}{f} \right) v = -g\left(\frac{\eta_x}{i\sigma} + \frac{1}{f}\eta_y \right)$$

$$\frac{f^2 i - \sigma^2 i}{\cancel{f\sigma}} v = -g\left(-\frac{i\eta_x f + \sigma\eta_y}{\cancel{f\sigma}} \right)$$

$$\cancel{i}\,(f^2-\sigma^2)\, v = -g\left(\frac{-if\eta_x}{i} + \frac{\sigma\eta_y}{i} \right)$$

$$(f^2-\sigma^2)\, v = -g\left(-f\eta_x - i\sigma\eta_y \right)$$

$$\boxed{v = \frac{-g}{(\sigma^2-f^2)}\left(f\eta_x + i\sigma\eta_y \right)}$$

$$-i\sigma u - fv = -g\eta_x$$

$$u = (-g\eta_x + fv) \Big/ (-i\sigma)$$

$$= \frac{i}{\sigma}\left(-g\eta_x + fv\right)$$

$$= -\frac{ig\eta_x}{\sigma} + \frac{if}{\sigma}\underbrace{\left[\frac{-g}{(\sigma^2-f^2)}\left(f\eta_x + i\sigma\eta_y\right)\right]}_{v}$$

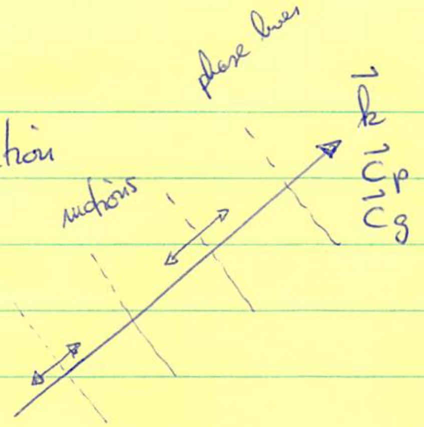
$$= \frac{-ig\eta_x(\sigma^2-f^2) - igf^2\eta_x - igf\,i\sigma\eta_y}{\sigma(\sigma^2-f^2)}$$

$$= \frac{-ig\sigma^{\cancel{2}}\eta_x \cancel{+igf^2\eta_x} \cancel{-igf^2\eta_x} - igf\,i\cancel{\sigma}\eta_y}{\cancel{\sigma}(\sigma^2-f^2)}$$

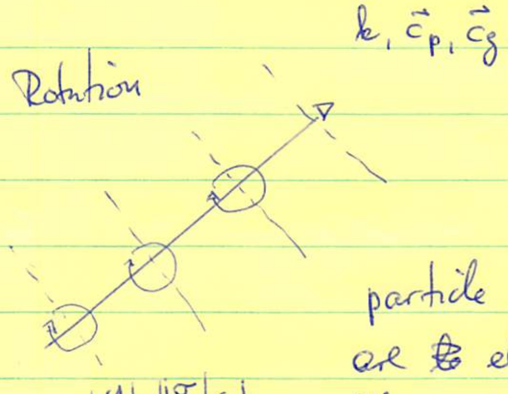
$$= (-ig\sigma\eta_x + gf\eta_y) \Big/ (\sigma^2-f^2)$$

$$\boxed{u = \frac{g}{(\sigma^2-f^2)}\left[f\eta_y - i\sigma\eta_x\right]}$$

No Rotation



Rotation



$$\left|\frac{u}{v}\right| = \left|\frac{i\sigma}{f}\right| < 1$$

particle motion are ~~to~~ ellipse with major axis along $\vec{k}$

Rotation makes these waves dispersive

$$c_{gx} = \frac{gD}{\sigma} k \qquad c_{gy} = \frac{gD}{\sigma} l$$

already discussed these waves during internal wave discussions p. 85/86

$N = N(z)$

$$\sigma^2 = \frac{N^2}{(n\pi / kD)^2} + f^2 \qquad \text{internal waves}$$

$$\sigma^2 = gD\,(k^2 + l^2) + f^2$$

$$\sigma^2 = gD_n \, k^2 + f^2$$

$$\sigma^2 = \boxed{\frac{N^2 D^2}{(n\pi)^2}} \cdot k^2 + f^2$$

$$D_n = \frac{N^2 D^2}{n^2 \pi^2} \cdot \frac{1}{g}$$

Use same dispersion curve graph to discuss

$$\sigma^2 = gD\,k^2 + f^2$$

$$\sigma = \cancel{ep}\left(gD\,k^2 + f^2\right)^{1/2}$$

$$\frac{\partial \sigma}{\partial k} = \frac{1}{\cancel{2}}\left(gD\,k^2+f^2\right)^{-1/2} \cdot \cancel{2}gD \cdot k$$

$$= gD \cdot k \Big/ \left(gD\,k^2+f^2\right)^{1/2}$$

$$c_{gx} = \frac{gD \cdot k}{\sigma} =$$

$$\vec{c}_p = \frac{\sigma}{(k^2+l^2)} \cdot \vec{k} \qquad c_{px} = \frac{\sigma}{(k^2+l^2)} \cdot k = \frac{\left(gD\,k^2+f^2\right)^{1/2}}{k^2+l^2} \cdot k$$

$$c_{gx} = gD \cdot \frac{k}{\sigma} \cdot \frac{\cancel{k}}{\cancel{k}} \; \frac{k^2+l^2}{k^2+l^2} \; \frac{\cancel{k}}{k} = \frac{gD\,k^2}{k^2+l^2} \cdot \frac{1}{c_{px}}$$

$$c_{px} = \frac{\sigma \cdot k}{k^2+l^2}$$

$$c_{gx} = \frac{gD\,k^2}{k^2+l^2} \cdot \frac{1}{c_{px}}$$

$$\boxed{\sigma^2 - f^2 = gD\left(k^2 + l^2\right)}$$

If $\sigma \ll f$ ⇒ very low frequency $\frac{\partial}{\partial t} \to 0$

steady state

$$\boxed{\begin{aligned} -fv &= -g\eta_x \\ fu &= -g\eta_y \end{aligned}}$$

geostrophic motion

If $\sigma = f$ ⇒ $\sigma^2 - f^2 = 0$ ⇒ $k = 0$ and $l = 0$

$$\boxed{\begin{aligned} u_t - fv &= 0 \\ v_t + fu &= 0 \end{aligned}}$$

$$\boxed{\begin{aligned} u &= \cos\sigma t = \cos ft \\ v &= \sin\sigma t = \sin ft \end{aligned}}$$

inertial

oscillation in circles

SKIP

<u>Reflection of Sverdrup Wave from Wall</u> (Poincare Waves)

~~Continuity equation~~ $\cancel{-i\sigma\eta_x + f\eta_y = 0 \quad \text{at } x = 0}$

$$u = \frac{g}{\sigma^2 - f^2}\left(f\eta_y - i\sigma\eta_x\right) \qquad \text{(p.-96)}$$

$u = 0$ at $x = 0$

⇒

$$-i\sigma\eta_x + f\eta_y = 0 \qquad \text{at } x = 0$$

$\eta = a_i e^{ikx + ily} + a_r e^{-ikx + ily}$ satisfies boundary condition if

$$i\sigma(ika_i - ika_r) + f(ila_i + ila_r) = 0$$

$\Rightarrow \quad a_r = a_i \dfrac{\sigma k - i f l}{\sigma k + i f l}$

$f = 0 \Rightarrow a_r = a_i$ same amplitude, no phase shift

$f \neq 0$ angle of incidence $\tan^{-1}(l/k)$ still equal to angle of reflection,

BUT amplitude differ

$\Rightarrow$ phase shift upon reflection

standing waves normal to wall

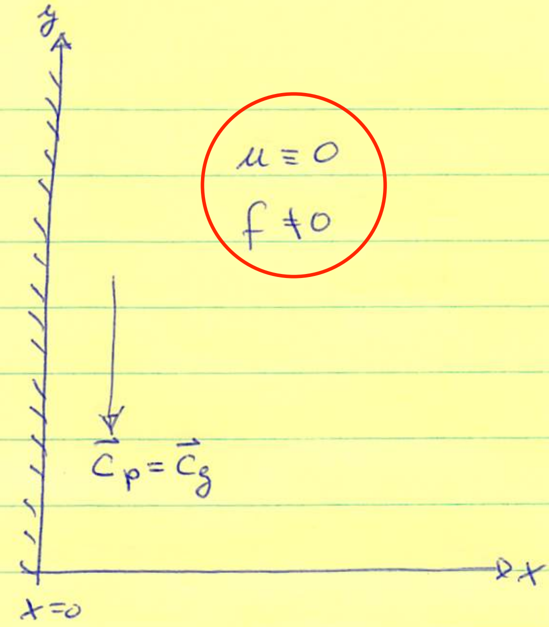
reflected waves with phase shift of waves ~~that~~ all travel along the wall.

As long as $\sigma > f$ they exist at any frequency and wave number

$\sigma^2 = f^2 + gD(k^2 + l^2)$

5.8 Kelvin Wave
(p. 121)

Solid Wall $x = 0$, $u \equiv 0$



$u \equiv 0$

$f \neq 0$

$-fv = -g\eta_x$ geostrophic across-shore

$v_t = -g\eta_y \quad \Big| \frac{\partial}{\partial y}$

$\eta_t + D v_y = 0 \quad \Big| \frac{\partial}{\partial t}$

$$\boxed{\eta_{tt} - gD\eta_{yy} = 0}$$

$e^{-i\sigma t}$

$-\sigma^2 - gD\eta_{yy} = 0$

pressure gradient along wall due to sealevel oscillations produce an acceleration along the wall, but pressure gradient along wall is in geostrophic balance with along-channel flow

$$\eta_{yy} + \frac{\sigma^2}{gD} = 0$$

choose $\boxed{\eta = a(x)\, e^{ily}}$ where $a(x)$ still unknown

$\Rightarrow \quad \sigma^2 = gD\, l^2$ same as shallow water wave without rotation

Kelvin Wave Dispersion Eq.

Combine $v = \frac{g}{f}\eta_x$ and $-i\sigma v = -g\eta_y$ from momentum

$\frac{g}{f}\eta_x = \frac{g}{i\sigma}\eta_y$ or $\boxed{-i\sigma\eta_x + f\eta_y = 0}$

$$\boxed{\eta = a(x) \cdot e^{ily}}$$

$$-i\sigma \eta_x + f\eta_y = 0$$

$$-\cancel{i}\sigma \frac{da}{dx} \cancel{e^{ily}} + f\cancel{i}l \cancel{e^{ily}} = 0$$

$$\boxed{\frac{da}{dx} = \frac{f}{\sigma} l} \quad \text{or} \quad \boxed{a(x) = a_0 e^{flx/\sigma}}$$

l/sigma = 1/c_phase
l/sigma = 1/sqrt(gD)

And the full solution is

$$\eta = a_0 \underbrace{e^{-i\sigma t + ily}} e^{flx/\sigma}$$

$$= a_0 e^{-i\sigma t \pm i\frac{\sigma y}{(gD)^{1/2}} \pm \frac{fx}{(gD)^{1/2}}}$$

Waves for $x > 0$
- finite solution as $x \to \infty$
- $\lim_{x \to \infty} \eta = 0$
- $l < 0$
- wave travels in the $-y$ direction here
- for $f > 0$ wave always travels with coast on its right