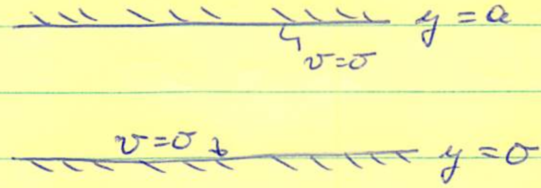
5.9
p.122

Wave guide modes



$$\nabla^2\eta + \frac{\sigma^2-f^2}{gD}\eta = 0 \qquad \text{with} \qquad i\sigma\eta_y + f\eta_x = 0$$

at $y=0,a$

$$v = -\frac{g}{\sigma^2-f^2}\left(f\eta_x + i\sigma\eta_y\right) \quad \text{from p.96}$$

Look for solutions $\eta = e^{ikx}\left[\underline{\cos\left(\frac{m\pi y}{a}\right) + \alpha_m \sin\left(\frac{m\pi y}{a}\right)}\right]$ *

⇓

Dispersion:

$$\boxed{k^2 = \frac{\sigma^2-f^2}{gD} - \underline{\left(\frac{m\pi}{a}\right)^2}} \qquad \boxed{\sigma > f}$$

Dispersion is the important part !

And the BC are at $y=0$ and $y=a$

$$i\sigma\frac{m\pi}{a}\left[-\sin\left(\frac{m\pi y}{a}\right) + \alpha_m\cos\left(\frac{m\pi y}{a}\right)\right]$$

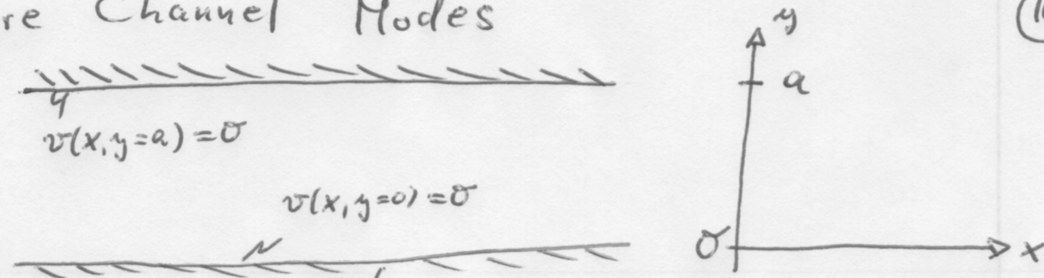
$$+ ifk\left[\cos\left(\frac{m\pi y}{a}\right) + \alpha_m \sin\left(\frac{m\pi}{a}\right)\right] = 0$$

⇓

$$\alpha_m = -\frac{f}{\sigma}\frac{ka}{m\pi} \qquad m = 1, 2, \ldots$$

(*) This allows or accomodates a phase shift.

Poincare Channel Modes



From p. 96 :
$$u = \frac{g}{\sigma^2-f^2}\left(f\eta_y - i\sigma\eta_x\right)$$

$$v = -\frac{g}{\sigma^2-f^2}\left(f\eta_x + i\sigma\eta_y\right)$$

$$\boxed{\nabla^2\eta + \frac{\sigma^2-f^2}{gD}\,\eta = 0}$$ with $v=0$ @ $y=0$ and $y=a$

or $i\sigma\eta_y + f\eta_x = 0$

Try $\eta(x,y) = e^{ikx}\left[\cos\left(\frac{m\pi y}{a}\right) + \alpha_m \sin\left(\frac{m\pi y}{a}\right)\right]$

$\eta_x = ik\,[\,..\,]$ $\qquad \eta_{xx} = -k^2\,[\,\ldots\,]$

$$\eta_y = e^{ikx}\left[\frac{m\pi}{a}(-)\sin(\ldots) + \frac{m\pi}{a}\alpha_m\cos(\ldots)\right]$$

$$\eta_{yy} = e^{ikx}\left[\left(\frac{m\pi}{a}\right)^2(-)\cos(\ldots) + \left(\frac{m\pi}{a}\right)^2\alpha_m(-)\sin(\ldots)\right]$$

$$-k^2\,\cancel{[\,\ldots\,]} - \left(\frac{m\pi}{a}\right)^2\cancel{\left[\cos(\ldots) + \alpha_m\sin(\ldots)\right]} + \frac{\sigma^2-f^2}{gD}\cancel{[\,\ldots\,]} = 0$$

$$\boxed{k^2 = \frac{\sigma^2-f^2}{gD} - \left(\frac{m\pi}{a}\right)^2}$$ Dispersion Relation $m=1,2,\ldots$

$k^2=0$ gives $\frac{\sigma^2-f^2}{gD} = \left(\frac{m\pi}{a}\right)^2$ or $m = \sqrt{\frac{(\sigma^2-f^2)}{gD}\cdot\frac{a^2}{\pi^2}}$

So if k^2 < 0, the solution decays exponentionally.

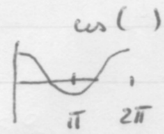
BC

$$i\sigma \eta_y + f \eta_x = 0$$

$$i\sigma \cancel{e^{ikx}} \left(\frac{m\pi}{a}\right)\left[-\sin(\dots) + \alpha_m \cos(\dots)\right] + f i k \cancel{e^{ikx}}\left[\dots\right] = 0 \quad @$$

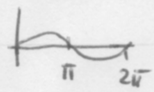
at $y=0$:

$$i\sigma\left(\frac{m\pi}{a}\right)\left[0 + \alpha_m\right] + f i k\left[1 + \alpha_m \cdot 0\right] = 0$$



at $y=a$

$$\left[0 \pm \alpha_m\right] + f i k\left[\pm 1 + \alpha_m \cdot 0\right] = 0$$



$\downarrow$

$$i\sigma\left(\frac{m\pi}{a}\right)\alpha_m + f i k = 0 \quad @ \quad y = 0, a$$

$\downarrow$

$$\alpha_m = -\frac{fk}{\sigma}\frac{a}{m\pi} = \frac{f}{c_p}\left(\frac{a}{m\pi}\right) \quad m = 1, 2, 3, \dots$$

not that important.

$$\cancel{\text{or } \frac{a}{m\pi} = -\alpha_m \frac{\sigma}{fk} \qquad \left(\frac{a}{m\pi}\right)^2 = \alpha_m^2 \frac{\sigma^2}{f^2 k^2}}$$

$$\boxed{k^2 = \frac{(\sigma^2 - f^2)}{gD} - \left(\frac{m\pi}{a}\right)^2}$$

and thus $\dfrac{\sigma^2 - f^2}{gD} > \left(\dfrac{m\pi}{a}\right)^2$ for oscillatory motions

<u>oscillatory motions for</u>

(1) <u>$\sigma > f$</u> <u>and</u> (2) <u>$m >$</u> $\left(\dfrac{\sigma^2 - f^2}{gD}\right)^{1/2} \cdot \dfrac{a}{\pi}$

because only then do we have $k^2 \geq 0$

$$k^2 = \frac{\sigma^2 - f^2}{gD} - \left(\frac{m\pi}{a}\right)^2 \qquad \sigma^2 = \left[k^2 + \left(\frac{m\pi}{a}\right)^2\right]\cdot gD + f^2$$

$$c_p^2 = \frac{\sigma^2}{k^2} = \frac{\sigma^2}{\sigma} \; \frac{a^2\, gD}{a^2(\sigma^2 - f^2) - m^2\pi^2\, gD}$$

$$= \left(\frac{k^2 a^2 + m^2\pi^2}{a^2}\right)\cdot \frac{gD}{k^2} + \frac{f^2}{k^2}$$

$$= \frac{(ka)^2 + (m\pi)^2}{(ak)^2}(gD) + \frac{f^2}{k^2}$$

$$= gD + \left(\frac{m\pi}{ak}\right)^2 gD + \frac{f^2}{k^2}$$

$$= gD\left[1 + \left(\frac{m\pi}{ak}\right)^2\right] + \frac{f^2}{k^2}$$

$\boxed{\text{always dispersive even for } f = 0\ !}$

$$\alpha_m = -\frac{f}{\sigma}\ \frac{k\,a}{m\pi} \qquad m = 1, 2, \ldots$$

$$\cancel{\text{or}\quad k = -\alpha_m \frac{\sigma}{f}\frac{\pi}{a}\cdot m}$$

$$\cancel{k^2 = \left(\alpha_m \frac{\sigma}{f}\frac{\pi}{a}\right)^2 \cdot m^2}$$

Dispersion:

$$\boxed{k^2 = \frac{\sigma^2 - f^2}{gD} - \left(\frac{m\pi}{\underline{a}}\right)^2}$$

As m gets larger k gets smaller and eventually imaginary

$$\Downarrow \quad m = 1, 2, \ldots < \left(\frac{\sigma^2 - f^2}{gD}\ \frac{a^2}{\pi^2}\right)^{1/2}$$

and $\sigma > f$

Otherwise, all "modes" are decaying ($\sigma < f$ and $m > \ldots$)

$\Rightarrow$ Poincaré Channel Modes

Regular Kelvin waves are possible, too, that is

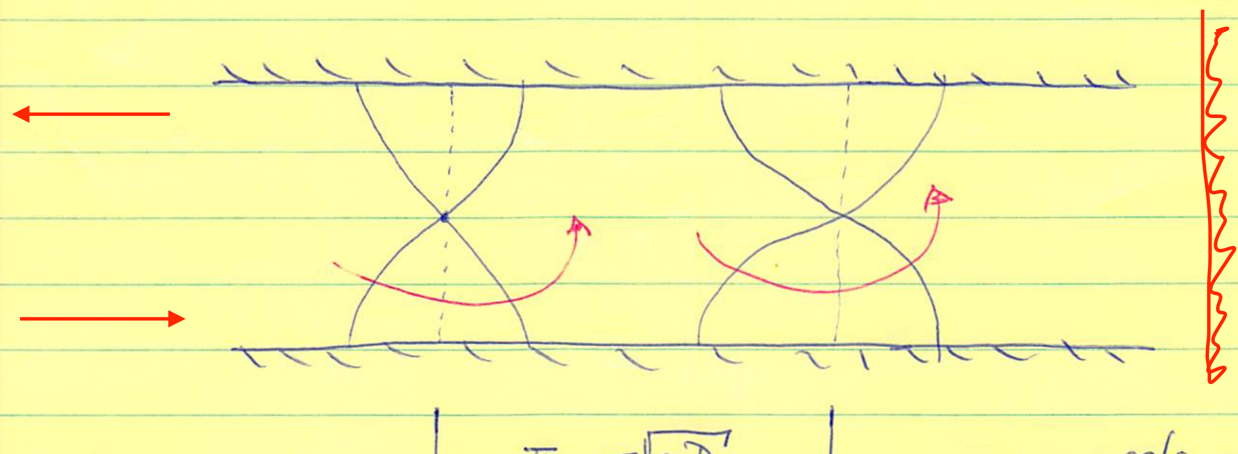
$$\eta = e^{-i\sigma t + i\sigma x/\sqrt{gD} - fy/\sqrt{gD}}$$

that is, a Kelvin wave moving east along $y = 0$
and also, — " — west — " — $y = 0$:

$$\eta = e^{-i\sigma t - i\sigma x/\sqrt{gD} + fy/\sqrt{gD}}$$

Because each Kelvin wave decays exponentially in y

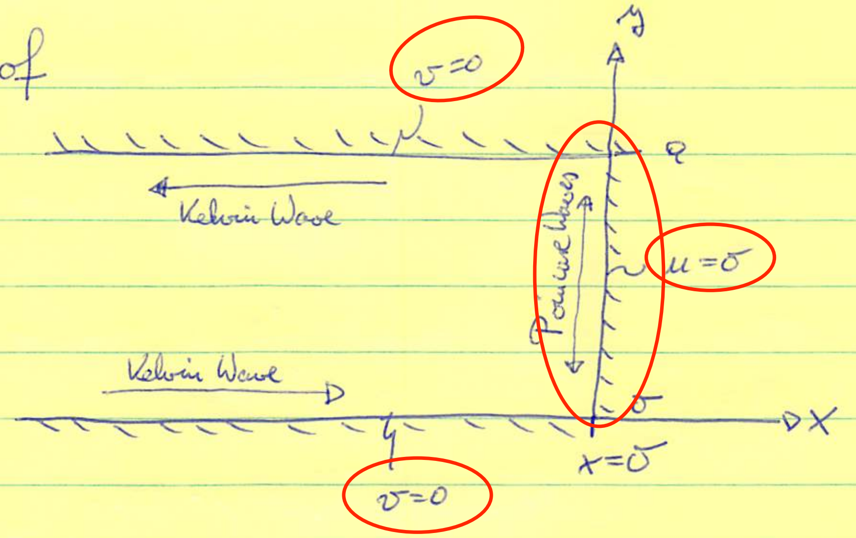
the two waves propagating in opposite direction do not form a standing oscillation, but rather motions that appear to rotate around "amphidromic points" where $\eta = 0$ at all times :



$$\frac{\pi}{k} = \frac{\pi\sqrt{gD}}{\sigma}$$

crests propagate once about each amphidromic point in period $\frac{2\pi}{\sigma}$

Discussion of



Kelvin waves have $v=0$ everywhere

include infinite series of Poincare Waves for which

$v=0$ at $y=0, a$

~~and~~ with

u = sum of all Poincare waves $\quad u = -u_{Kelvin}$ at $x=0$

Perfect Kelvin Wave reflection at a closed channel has
all Poincare wave modes decay away from the reflecting wall.

1. $\sigma^2 < f^2$ $\quad\to$ all Poincare Waves decay as $x \to -\infty$
 so solution away from $x=0$ are Kelvin

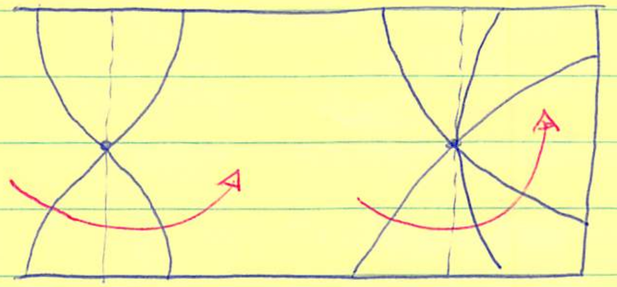
2. $\sigma^2 > f^2$ with $m_{max} = \left(\dfrac{\sigma^2 - f^2}{gD} \dfrac{a^2}{\pi^2} \right)^{1/2} > 1$

 $\to$ one or more Poincare Waves vary with x such that the reflected wave(s) is not a pure Kelvin wave, but a hybrid

$\to$ a sufficiently wide ("a" large enough) channel will allow perfect Kelvin wave reflection for

$$\frac{\sigma^2 - f^2}{gD} < \frac{\pi^2}{a^2}$$

3. $\sigma^2 < f^2$ or $\sigma^2 > f^2$ but "a" small



Taylor, G. J., 1920: Tidal oscillations in gulfs and rectangular basins. Proc. Roy. Soc., 20, 148-181.

Tides in embayments such as

North-Sea
Irish Sea
Gulf of California
Adriatic Sea etc., etc.