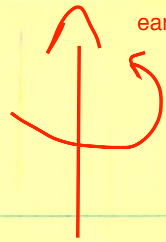


earth rotational vector

at North Pole the normal component of the earth's rotational vector is f=+fmax

Equator normal component of the earth's rotational vector is f=0

at South pole the normal component of earth's rotational vector is f=-fmax

Rossby Waves

Vorticity Waves due to (a) $f = f(y) = f_0 + \beta y$ Coriolis varies with latitude

beta-plane aproximation

(b) $D \neq D(x,y)$ Depth varies

Consider shallow water equations

$$\boxed{\begin{aligned} (1)\quad & u_t - fv = -g\eta_x \\ (2)\quad & v_t + fu = -g\eta_y \end{aligned}}$$

$\Big|\cdot\frac{\partial}{\partial y}$ $\quad u_{ty} - \beta v - f v_y = -g\eta_{xy}$

$\Big|\cdot\frac{\partial}{\partial x}$ $\quad v_{tx} + f u_x = -g\eta_{yx}$

Take the curl of the momentum to derive the vorticity equation [just for the vertical component of the vorticity vector].

(2)−(1):

$$\boxed{(v_x - u_y)_t + f(u_x + v_y) + \beta v = 0}$$

From continuity

$$\eta_t + (uD)_x + (vD)_y = 0$$

with ~~$D(y) = e^{-By/f}$~~ $D(y) = e^{-By/f}$ this continuity gives

$$(u_x + v_y) = -\frac{\eta_t}{D} - \frac{v}{D}(-)\frac{B}{f}D$$

and the vorticity equation becomes

$$\boxed{(v_x - u_y)_t = -\beta v - Bv + \frac{f}{D}\eta_t}$$

The "planetary β" has the same dynamical as the "topographic B" effect ?

$$\eta_t + (uD)_x + (vD)_y = 0 \qquad \text{Continuity}$$

$$\eta_t + u_x \cdot D + u D_x + v_y D + v D_y = 0$$

$$\eta_t + D(u_x + v_y) + \vec{u} \cdot \vec{\nabla} D = 0$$

$$(u_x + v_y) = \frac{-\eta_t}{D} - \frac{\vec{u}}{D} \nabla D$$

into (2) − (1)

$$(v_x - u_y)_t + \frac{f}{D}(-)\eta_t + \frac{f}{D}(-)\vec{u}\,\nabla D + \beta v = 0$$

with $D = D(y) = e^{-By/f}$ $\qquad D_y = -\frac{B}{f} e^{-By/f}$

$$(v_x - u_y)_t - \frac{f}{D}\eta_t - \frac{\cancel{f}}{\cancel{D}} v (-)\frac{B}{\cancel{f}} \cdot \cancel{D} + \beta v$$

$$(v_x - u_y)_t - \frac{f}{D}\eta_t + Bv + \beta v = 0$$

Let focus on "planetary - β"

$D = \text{const.}$

horizontally non-divergent ocean, e.g., $u_x + v_y = 0$ $(\eta_t = 0)$

[very low-frequency $\eta_t \to 0$]

D/T << D U/L or
f << U/L or
f*U/L << 1 or
Rossby Number << 1

The vorticity equation then becomes

$$\boxed{\underbrace{(v_x - u_y)_t}_{\text{relative vorticity}} + \underbrace{\beta v}_{\text{planetary vorticity}} = 0}$$

non-divergent flow is $u = -\psi_y$ and $v = \psi_x$ stream function

$$\boxed{\nabla^2 \psi_t + \beta \psi_x = 0}$$

What does this say about the relation of velocity (u,v) and wave number vector (k,l)?

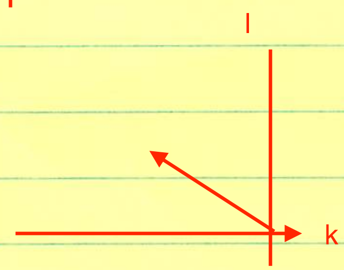
u —> -i*l
v —> +i*k
|u/v| = | l/k |

Plane wave solutions

$$\psi = e^{-i\sigma t + ikx + ily}$$

give dispersion

$$\boxed{\sigma = \frac{-\beta k}{k^2 + l^2}}$$



which we can write as

$$\boxed{\left(k + \frac{\beta}{2\sigma}\right)^2 + l^2 = \left(\frac{\beta}{2\sigma}\right)^2}$$

which in the (k, l) plane looks like

$$\left(k+\frac{\beta}{2\sigma}\right)^2 + l^2 = \left(\frac{\beta}{2\sigma}\right)^2$$

$$k^2 + 2\frac{k\beta}{2\sigma} + \cancel{\left(\frac{\beta}{2\sigma}\right)^2} + l^2 = \cancel{\left(\frac{\beta}{2\sigma}\right)^2}$$

$$k^2 + l^2 = -k\frac{\beta}{\sigma}$$

$$\boxed{\sigma = -k\frac{\beta}{k^2+l^2}}$$

$$k^2 + l^2 = -k\frac{\beta}{\sigma}$$

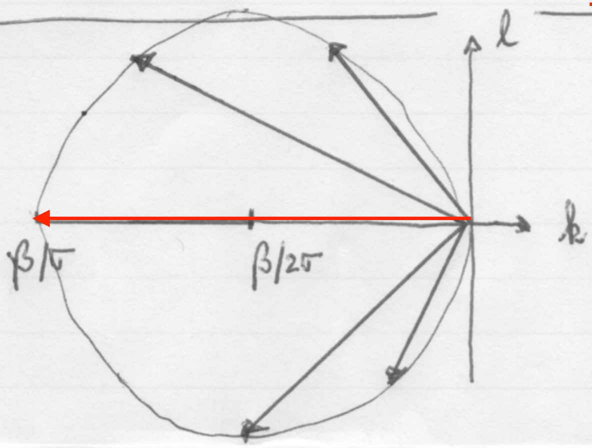
$$k^2 + l^2 + 2k\frac{\beta}{2\sigma} + \left(\frac{\beta}{2\sigma}\right)^2 - \left(\frac{\beta}{2\sigma}\right)^2 = 0$$

$$k^2 + 2k\frac{\beta}{2\sigma} + \left(\frac{\beta}{2\sigma}\right)^2 + l^2 = \left(\frac{\beta}{2\sigma}\right)^2$$

$$\boxed{\underline{\left(k+\frac{\beta}{2\sigma}\right)^2} + \underline{l^2} = \underline{\left(\frac{\beta}{2\sigma}\right)^2}}$$

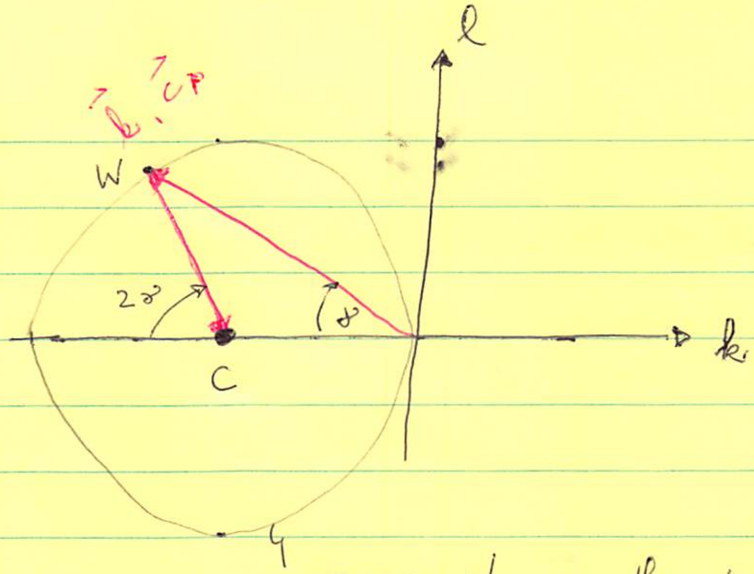
$$\boxed{a^2+b^2=c^2}$$

which wave numbers (k,l) are on this circle



That's a circle of radius $\beta/2\sigma$

displaced on the k-axis by $\beta/2\sigma$ to the left!



$\sigma = \text{const}$ on the circle with radius $|WC| = \left|\dfrac{\beta}{2\sigma}\right|$

The center of the circle C
is at $k = -\beta/2\sigma$
$l = 0$
with radius $\beta/2\sigma$

For $l = 0$ & $\sigma = -\beta/k$
$c_p = \sigma/k = -\beta/k^2$

- Phase speed is always to the West, but can have either north- or south-ward components

$$c_{px} = \frac{\sigma}{k} = -\beta\Big/ k^2+l^2$$

$$c_{py} = \frac{\sigma}{l} = -\beta k \Big/ l(k^2+l^2)$$

This type of waves is called Rossby Waves.
(They more fundamentally emerge as the "driver" of the non-divergent and degenerate geostrophic equations at $Ro = U/fL \ll 1$

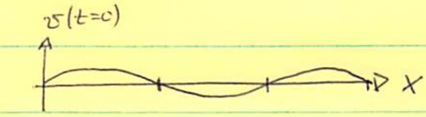
Physics of Rossby Waves

Assume $\frac{\partial}{\partial y} \ll \frac{\partial}{\partial x}$ $\Rightarrow$ $v_{xt} + \beta v = 0$

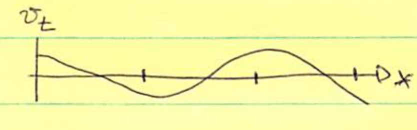
relative + planetary vorticity = 0

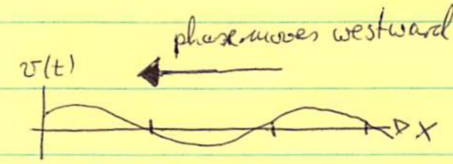
North-South motions v change the local (relative) vorticity v_x

$$v(x, t=0) = \sin(kx)$$



$$\frac{\partial}{\partial t}\left(v(x,t)\right) = -\int \beta v \, dx \quad \propto \cos(kx)$$





$$v(x,t) = \underbrace{v(x, t=0)}_{\sin(kx)} + \Delta t \underbrace{\frac{\partial v}{\partial t}}_{\cos(kx)}$$

$\kappa^2 = k^2 + l^2$

$$c_{gx} = \frac{\partial \sigma}{\partial k} = -\frac{\beta}{\kappa^2} + \frac{2\beta k^2}{\kappa^4} = \frac{\beta \kappa^2(-\sin^2\gamma + \cos^2\gamma)}{\kappa^4} = \beta \frac{\cos(2\gamma)}{\kappa^2}$$

cos(2a) = cos(a)*cos(a) - sin(a)*sin(a)

$$c_{gy} = \frac{\partial \sigma}{\partial l} = \frac{2\beta k l}{\kappa^4} = -\beta \frac{\sin(2\gamma)}{\kappa^2}$$

Discuss dispersion relation from sketch on p.-109

- a mostly west-ward phase $\vec{c}_p$ propagates energy mostly eastward
- a more north-ward phase propagates energy mostly westward
- the velocity vector (u, v) is NORMAL to wavenumber k

$u_x + v_y = 0$

$\begin{pmatrix} k \\ l \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = 0$

$v = \psi_x = ik\psi$ and $u = -\psi_y = -il\psi$