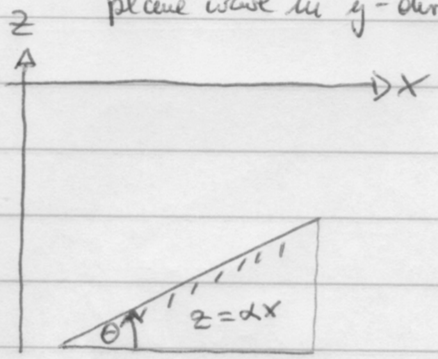
Bottom-Trapped Waves

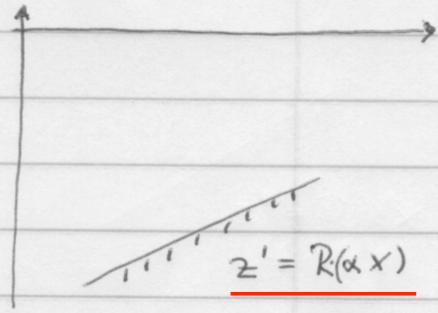
$$R^2 = \frac{N^2}{f^2-\sigma^2}$$

plane wave in y-direction e^{ily}



Co-ordinate Transformation

$z' = R \cdot z$



…ion: $\quad p_{xx} + p_{yy} + \frac{f^2-\sigma^2}{N^2} p_{zz} = 0 \qquad\qquad p_{xx} - l^2 p + p_{z'z'} = 0$

…ry
…ion: $\quad \underbrace{i\sigma(f^2-\sigma^2)p_z}_{w} = \alpha N^2 \underbrace{(i\sigma p_x - f p_y)}_{u} \qquad\qquad p_{z'} = R\alpha\left(p_x - \frac{lf}{\sigma} p\right.$

at $z = \alpha x$ where $w = \alpha u$ $\qquad\qquad$ at $z' = R\alpha x$

problem simplifies when we consider a "new slope" $R\alpha$

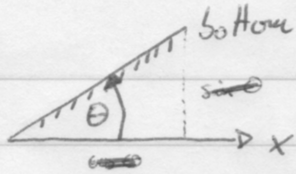
Now "new slope"

$$R \cdot \alpha = \frac{N \cdot \alpha}{\sqrt{f^2-\sigma^2}} = \frac{N\cdot\alpha / f}{\sqrt{1-\omega^2}} = \frac{S'}{\sqrt{1-(\sigma/f)^2}}$$

$S' = N \cdot \alpha / f$ and $\omega = \sigma / f$

↳ strong stratification "appears" as an "effective" steep bottom slope

As $S' \to \infty$, the bottom "appears" as a vertical wall.



$z' = R\alpha \cdot x$

$$\theta = \tan^{-1}(R\alpha)$$

or $R\alpha = \tan\theta$ or $R\alpha\cos\theta = \sin\theta$

Thus we can write the solutions as

$$p(t,x,z') = \exp\Big[-i\sigma t + i l y \pm i k (x\cos\theta + z'\sin\theta)$$

tical decay"

$$- i m (z'\cos\theta - x\sin\theta)\Big]$$

k is the wave number parallell to the bottom and m is the wave number perpendicular to the bottom

$$\boxed{m^2 = k^2 + l^2}$$

results if we place this solution into the wave eq.

and the bottom boundary condition gives us

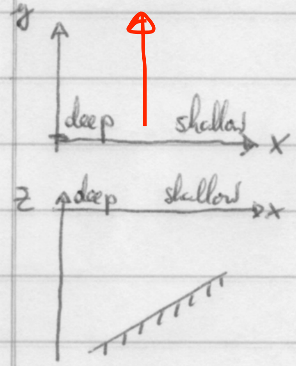
$$m = \frac{l}{\omega}\left[\frac{S^2}{(1-\omega^2+S^2)}\right]^{1/2} \stackrel{?}{=} \frac{l}{\omega}\sqrt{\sin^2\theta}$$

$$k = \frac{l}{\omega}\left[\frac{(S^2-\omega^2)(1-\omega^2)}{(1-\omega^2+S^2)}\right]^{1/2} \stackrel{?}{=} \frac{l}{\omega}\sqrt{\sin^2\theta-\omega^2}$$

Some observations

(1) For decay away from the bottom $(m > 0)$

l and ω must have the same sign



$\omega > 0$ & $l > 0$

& propagates with shallow water on the right $(f > 0)$

(2) For $\omega = \frac{\sigma}{f} < 1$ m is always real and motions are always bottom-trapped

(3) Waves propagate along the bottom (k real) as long as $\omega < S'$

(4) If $\omega > S'$ then waves decay exponentially along bottom

& if $S' > 1$, then the waves always propagate because $\omega < 1$

(5) As S' becomes larger, the waves become more trapped near bottom

(6) A S' becomes small $\to 0$, evanescent.

(7) As $\omega \to 0$, both k and m become large

that is short waves trapped near bottom

S is Slope Burger Number:

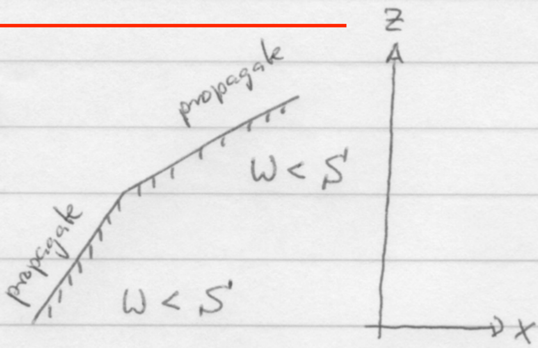
S = N*alpha/f

$$\omega = \frac{\sigma}{f} < S'$$

propagates along the bottom and encounters a different bottom slope

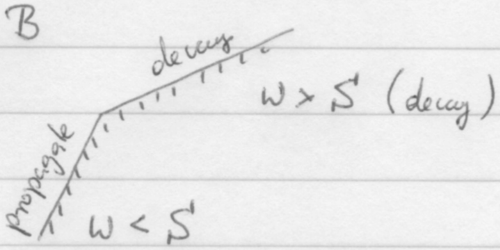
sigma < N*alpha

Case-A



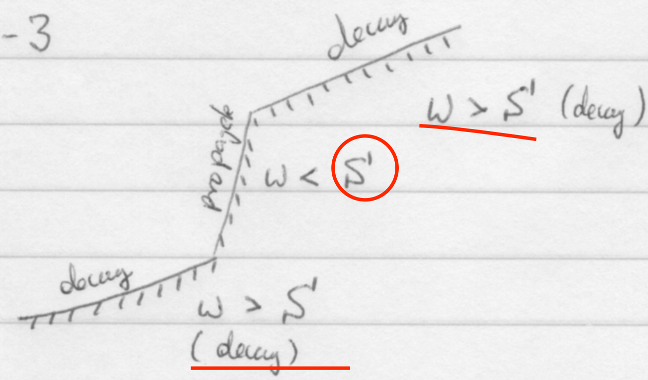
Waves propagate along slope as long as $\omega < S'$

Case-B



When $\omega > S'$ on the new slope, then the wave must reflect

Case-3



Which leads to "trapping" if the $\omega < S'$ region is bordered by two regions with $\omega > S'$ that keep reflecting the wave thus keeping it over the steep slope?

$$w = \alpha v \qquad \text{at } z = \alpha x$$

$$\overbrace{i\sigma(f^2-\sigma^2)\,p_z}^{w} = \alpha\, \overbrace{N^2(i\sigma p_x - f p_y)}^{v}$$

$p_y = i l p$ wave oscillations in along-slope "y" direction

$$p_z = \frac{\alpha N^2}{f^2-\sigma^2} \cdot \frac{1}{i\sigma}\left(i\sigma p_x - i f p\right)$$

$$p_z = \alpha R^2 \left(p_x - \frac{f}{\sigma} p\right)$$

$$z' = R\cdot z \quad \Rightarrow \quad \frac{\partial\,\cdot}{\partial z} = \frac{\partial\,\cdot}{\partial z'} \cdot \frac{\partial z'}{\partial z} = R\frac{\partial}{\partial z'}$$

example.

$$p_z = R\cdot p_{z'}$$

and

$$R\, p_{z'} = \alpha R^2\left(p_x - \frac{fl}{\sigma} p\right) \qquad \text{at } z' = R\alpha x$$

$$p_{z'} = \alpha R\left(p_x - \frac{fl}{\sigma} p\right) \qquad \text{at } z' = R\alpha x$$

$$\Theta = \tan^{-1}(R\alpha) \qquad \frac{\sin R\alpha}{\cos R\alpha} = \tan(R\alpha)$$

$$p_{xx} - l^2 p + p_{z'z'} = 0$$

$p_x \quad ik\cos\theta + m\sin\theta \qquad p_{z'} = ik\sin\theta - m\cos\theta$

$p_{xx} \quad -k^2\cos^2\theta + m^2\sin^2\theta \qquad p_{z'z'} = -k^2\sin^2\theta + m^2\cos^2\theta$

$$-k^2\underline{\cos^2}\theta + m^2\underline{\sin^2}\theta - l^2 - k^2\underline{\sin^2}\theta + m^2\cos^2\theta = 0$$

$$-k^2(\sin^2\theta + \cos^2\theta) + m^2(\sin^2\theta + \cos^2\theta) = l^2$$

$$(k^2 + l^2) = m^2\,\cancel{(\sin^2\theta + \cos^2\theta)}$$

$$p_{z'} = R\alpha\left(p_x - \frac{lf}{\sigma}p\right) \quad \text{at} \quad z' = R\alpha x$$

$$(ik\sin\theta - m\cos\theta)\cancel{p} = R\alpha\left(ik\cos\theta + m\sin\theta - \frac{lf}{\sigma}\right)p$$

$$\frac{ik}{R\alpha}\underbrace{(\sin\theta - R\alpha\cos\theta)}_{\sin\theta} = \frac{m}{R\alpha}(\cancel{\cos\theta} + {}^{R\alpha}\!\!\!\!\!\!\!\!\quad \sin\theta) - \cancel{R\alpha}\frac{f}{\sigma}l$$

$$\cancel{\frac{ik}{R\alpha}}\ 1 \quad 0 = m(1$$

$\omega = \sigma/f$

$S = \dfrac{N\cdot\alpha}{f}$

$R\alpha = \dfrac{S}{(1-\omega^2)^{1/2}}$

$$i k \sin\theta - m\cos\theta - R\alpha i k\cos\theta + R\alpha m\sin\theta + R\alpha \frac{lf}{\sigma} = 0$$

$$m^2 = k^2 + l^2$$

$$\cancel{i k \left(\sin\theta - \underbrace{R\alpha\cos\theta}_{\sin\theta}\right)} - m(\cos\theta + R\alpha\sin\theta) + R\alpha\frac{lf}{\sigma} = 0$$

$$\cancel{i k = m(\cos\theta + R\alpha\sin\theta) - R\alpha\frac{lf}{\sigma}}$$

$$\left(\cancel{\frac{k}{m}}\right) =$$

$$m(\cos\theta + R\alpha\sin\theta) = R\alpha \quad \frac{lf}{\sigma} \qquad \Big/ \cdot \cos\theta$$

$$m\left(\cos^2\theta + \underbrace{R\alpha\cos\theta}_{\sin\theta}\sin\theta\right) = \underbrace{R\alpha\cos\theta}_{\sin\theta} \quad \frac{lf}{\sigma}$$

$$m(\cos^2\theta + \sin^2\theta) = \frac{lf}{\sigma}\sin\theta$$

$$\boxed{m = \frac{lf}{\sigma}\sin\theta} \qquad \text{or} \quad m^2 = \left(\frac{l}{\omega}\right)^2 \sin^2\theta$$

$$\boxed{m = \frac{l}{\omega} \sin\theta}$$

$$k^2 = m^2 - l^2$$

$$= \frac{l^2}{\omega^2} \sin^2\theta - l^2 =$$

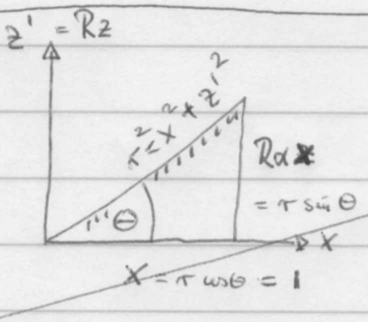
$$= \frac{l^2}{\omega^2}\left(\sin^2\theta - \omega^2 \right)$$

$$\boxed{k_r = \frac{l}{\omega}\sqrt{\sin^2\theta - \omega^2}}$$

$$\sin\theta = R\alpha \cos\theta$$

$$\tan\theta = R\alpha = \frac{N}{\sqrt{f^2 - \sigma^2}}\alpha = \frac{N\cdot\alpha/f}{\sqrt{1-\omega^2}} = \frac{S'}{\sqrt{1-(\sigma/f)^2}}$$

$\alpha)^2 =$



$$\cancel{\tan\theta = \frac{R\alpha x}{x} = R\alpha}$$

$r = 1 \quad r^2 = 1$

$$\cancel{r^2 = x^2 + R^2\alpha^2 x^2 = x^2(1 + R^2\alpha^2)}$$

$$x^2 + z'^2 = 1 = \cos^2\theta + \sin^2\theta$$

$$\left(\frac{S^2}{1-\omega^2+S^2}\right)^{1/2} \overset{?}{\doteq} \sin\Theta \quad = R\alpha\cos\Theta = \cos\Theta \quad \frac{S'}{\sqrt{1-\omega^2}}$$

$$\boxed{\frac{S^2}{1-\omega^2+S^2} \overset{?}{\doteq} \sin^2\Theta} = 1-\cos^2\Theta$$

$$R\alpha = \frac{S'}{\sqrt{1-\omega^2}} = \tan\Theta = \frac{\sin\Theta}{\cos\Theta} = R\alpha$$

$$R\alpha\cos\Theta = \sin\Theta$$

$$(S^2-\omega^2)(1-\omega^2) = S^2 - \omega^2 S^2 - \omega^2 + \omega^4$$

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need to prove this

$$\frac{N^2\alpha^2/f^2}{1-\frac{\omega^2}{f^2}+N^2\alpha^2/f^2} \doteq \sin^2\Theta$$

$$R^2 = \frac{N^2}{f^2-\sigma^2}$$

$$\frac{N^2\alpha^2/f^2}{1-\frac{1}{f^2}\left(\sigma^2+N^2\alpha^2\right)} = \frac{N^2\alpha^2}{f^2-\sigma^2+N^2\alpha^2} = \frac{\alpha^2}{(f^2-\sigma^2)/N^2+\alpha^2}$$

$$\alpha^2R^2/(1+\alpha^2R^2) = \alpha^2/\left(\frac{1}{R^2}+\alpha^2\right)$$